Metóda Lagrangeových multiplikátorov:

$f: \mathbb{R}^2 \to \mathbb{R}$, $\boxed{M: F(x,y)=0}$

$$L(x,y,\lambda) = f(x,y) - \lambda F(x,y)$$

1. Určíme stacionárne body L:

$$\left[\begin{array}{l} L'_x(x,y,\lambda) = f'_x(x,y) - \lambda F'_x(x,y) = 0 \\ L'_y(x,y,\lambda) = f'_y(x,y) - \lambda F'_y(x,y) = 0 \\ L'_\lambda(x,y,\lambda) = -F(x,y) = 0 \end{array}\right]$$

(x^*, y^*, λ^*)

$\underset{>0}{6}h_1^2 + \underset{>0}{6}h_2^2 = H(h_1,h_2)$ pozitívne definitná

$\underset{<0}{-h_1^2} \underset{<0}{-h_2^2}$ negatívne definitná

$h_1^2 - h_2^2$ indefinitná

2. Zostavíme Hessovu maticu L vzhľadom k premenným (x,y)

$L''_{xx} = ?$

$L''_{xy} = ?$

$L''_{yy} = ?$

Podľa Silvestrovho kritéria určíme definitnosť kvadratickej formy

$$d^2 L_{(h_1,h_2)}(x^*,y^*,\lambda^*) = (h_1, h_2) \begin{pmatrix} L''_{xx}(x^*,y^*,\lambda^*) & L''_{yx}(x^*,y^*,\lambda^*) \\ L''_{yx}(x^*,y^*,\lambda^*) & L''_{yy}(x^*,y^*,\lambda^*) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

- Pozitívne definitná kvadratická forma

  (x^*,y^*) je lok. minimum f na M (hlavné minory sú kladné!)

- Negatívne definitná kvadratická forma

  (x^*,y^*) je lok. maximum f na M

  (hlavné minory striedajú znamienko počínajúc záporným!)

3. Ak sme sa nerozhodli v 2. pokračujeme v nasledovnom:

$F'_x = ?$, $F'_y = ?$,

$$\boxed{F'_x(x^*,y^*)h_1 + F'_y(x^*,y^*)h_2 = 0}$$

vyjadríme si: $h_2 = \underbrace{-\dfrac{F'_x(x^*,y^*)}{F'_y(x^*,y^*)} h_1}_{\eta(h_1)}$, $F'_y(x^*,y^*) \neq 0$

derivácia implicitne zadanej funkcie

$\boxed{y = \eta(x): F(x, \eta(x)) = 0}$

a určíme

$$d^2 L_{(h_1,\eta(h_1))}(x^*,y^*,\lambda^*) = (h_1, \eta(h_1)) \begin{pmatrix} L''_{xx}(x^*,y^*,\lambda^*) & L''_{yx}(x^*,y^*,\lambda^*) \\ L''_{yx}(x^*,y^*,\lambda^*) & L''_{yy}(x^*,y^*,\lambda^*) \end{pmatrix} \begin{pmatrix} h_1 \\ \eta(h_1) \end{pmatrix} = \underbrace{\text{niečo}} \cdot h_1^2$$

obmedzíme sa na špecifický podpriestor dim 1.

>0 lok. min.

<0 lok. max.

Pr1 $f(x,y) = xy - x^2 - y^2 + x + y$ $\quad M: x + y - 4 = 0$

$$L(x,y,\lambda) = xy - x^2 - y^2 + x + y - \lambda(x+y-4)$$

$$\left.\begin{array}{l} L'_x = y - 2x + 1 - \lambda = 0 \\ L'_y = x - 2y + 1 - \lambda = 0 \end{array}\right\} \ominus \Rightarrow \quad y - x - 2(x-y) = 0 \Rightarrow \boxed{y = x}$$

$$L'_\lambda = -(x+y-4) = 0$$

$$2x = 4 \Rightarrow \boxed{x = 2} = y$$

$$\underline{\lambda} = 2 - 4 + 1 = \underline{-1}$$

$$\underline{P = [2,2]}$$

$$F(x,y) = x + y - 4 = 0 \qquad F'_x = 1, \; F'_y = 1$$

$$F'_x(P)h_1 + F'_y(P)h_2 = 0$$

$$h_1 + h_2 = 0 \Rightarrow \boxed{h_1 = -h_2}$$

$L''_{xx} = -2$

$L''_{yy} = -2$

$L''_{xy} = 1$

$$\underset{(h_1,h_2)}{dL^2(x,y,\lambda)} = (h_1\ h_2) \underbrace{\begin{pmatrix} \boxed{-2} & 1 \\ 1 & -2 \end{pmatrix}}_{H} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$\det H = 4 - 1 = 3 > 0$

$-2 < 0$

→ negative def. kvadratická forma · lok. maximum

$$= -2h_1^2 + 2h_1h_2 - 2h_2^2$$

$$= -2h_1^2 - 2h_1^2 - 2h_1^2 = -6h_1^2$$

nemusíme, lebo sme rozhodli vyššie

negatívne definitná kvadratická forma ⇒ lok. maximum P

(Ide aj dosadiť obmedzenie, ale horšie počítanie)

$\boxed{\text{CVIKO}}$

$\boxed{\text{CVIKO}}$

Pr2 $f(x,y,z) = x + 2y + 3z$ $\quad M: x^2 + y^2 = z$ – paraboloid

(ide aj dosadiť výpočet je obdobný)

$$\boxed{F(x,y,z) = x^2 + y^2 - z}$$

$$L(x,y,z,\lambda) = x + 2y + 3z + \lambda(x^2 + y^2 - z)$$

(vidíme, že pre $\lambda = 0$ nemá žiadny stac. bod t.j. bez obmedzenia)

$L'_x = 1 + \lambda 2x = 0 \Rightarrow x = -1/6$

$L'_y = 2 + 2y\lambda = 0 \Rightarrow y = -1/3$

$L'_z = 3 - \lambda = 0 \Rightarrow \lambda = 3$

$L'_\lambda = x^2 + y^2 - z = 0 \Rightarrow \frac{1}{36} + \frac{1}{9} = \frac{5}{36} = z$

$$P = [-1/6, -1/3, 5/36]$$

$L''_{xx} = 2\lambda$

$L''_{yy} = 2\lambda$

$L''_{zz} = 0$

$L''_{zy} = 0 = L''_{zx}$

$L''_{xy} = 0$

$$\underset{(h_1,h_2,h_3)}{d^2L(x,y,z,\lambda)} = (h_1\ h_2\ h_3) \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \overset{>0}{6}h_1^2 + \overset{>0}{6}h_2^2$$

$\underline{\det(H) = 0}$

(nezáleží na h_3)

pozitívne def. kvadratická forma

Lok. minimum v P_2

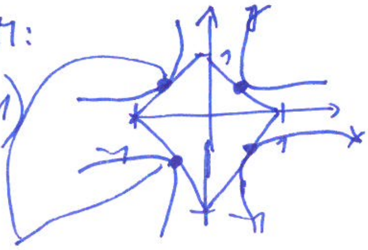
$F'_x = 2x, \; F'_y = 2y, \; F'_z = -1$

$$F'_x(P)h_1 + F'_y(P) \cdot h_2 + F'_z(P) \cdot h_3 = 0$$

$$-\frac{1}{3}h_1 - \frac{2}{3}h_2 - h_3 = 0 \Rightarrow h_3 = \frac{1}{3}h_1 + \frac{2}{3}h_2$$

Pr 3

$f(x,y) = x\cdot y$ $\quad x,y \neq 0 \leftarrow$ M: $|x|+|y|=1$



jednotková sféra v súčtovej metrike

vrstevnice $x\cdot y = c$

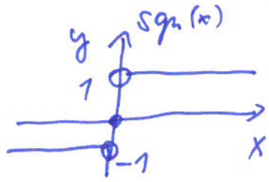
$L(x,y,\lambda) = x\cdot y - \lambda(\operatorname{sgn}(x)x + \operatorname{sgn}(y)\cdot y - 1)$

$L'_x = y - \lambda(\operatorname{sgn}(x)) = 0$

$L'_y = x - \lambda(\operatorname{sgn}(y)) = 0$

$L'_\lambda = -(|x|+|y|-1) = 0$

Zrejme lok. extrémy



$y = +\lambda \operatorname{sgn}(x)$

$x = \lambda \operatorname{sgn}(y)$

CVIKO

$1 = |\lambda \operatorname{sgn}(x)| + |\lambda \operatorname{sgn}(y)| = |\lambda| + |\lambda| = 2|\lambda| \Rightarrow |\lambda| = \frac{1}{2}$

$\lambda = \pm \frac{1}{2}$

stacionárne body:

- $\lambda = 1/2$

$y = \frac{1}{2}\operatorname{sgn}(x)$

$x = \frac{1}{2}\operatorname{sgn}(y)$

$P_1 = [-1/2, -1/2]$

$P_2 = [1/2, 1/2]$

$\lambda = -1/2$

$y = -\frac{1}{2}\operatorname{sgn}(x)$

$x = -\frac{1}{2}\operatorname{sgn}(y)$

$P_3 = [-1/2, 1/2]$

$P_4 = [1/2, -1/2]$

Prípadne AG nerovnosť:

$\frac{x+y}{2} \geq \sqrt{xy}$ pre $x,y > 0$ máme M: $x+y=1$

rovnosť nastáva $\Leftrightarrow x=y$

$\frac{1}{2} \geq \sqrt{xy}$

$\left(\frac{1}{2}\right)^2 = x^2 \Rightarrow x = \frac{1}{2} = y$ → zo symetrie dostaneme ostatné body

sedlový bod bez obmedzenia

$L''_{xx} = 0, \; L''_{yy} = 0, \; L''_{xy} = 1$

$\underset{(h_1,h_2)}{d^2L(x,y)} = (h_1\ h_2)\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{H}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ nevieme rozhodnúť, $\det H < -1$

$F(x,y) = \operatorname{sgn}(x)\cdot x + \operatorname{sgn}(y)\cdot y - 1$

$F'_x = \operatorname{sgn}(x) \quad F'_y = \operatorname{sgn}(y)$

$F'_x(P)\cdot h_1 + F'_y(P)\cdot h_2 = 0$

$\operatorname{sgn}(x)\cdot h_1 + \operatorname{sgn}(y)\cdot h_2 = 0$

$h_1 = -\operatorname{sgn}(x)\cdot\operatorname{sgn}(y) h_2$

$= 2h_1\cdot h_2 = -2\operatorname{sgn}(x)\operatorname{sgn}(y) h_2^2$

$\operatorname{sgn}(x) = \operatorname{sgn}(y) = \pm 1 \Rightarrow$

$\underset{h_1}{d^2L(x,y)}$ je negatívne def. kvadratická forma

$\Rightarrow$ lok. maximum.

$\operatorname{sgn}(x) \neq \operatorname{sgn}(y)$

pozitívne def. kvadr. forma

lok. minimum.

P_1, P_2 lok. maximum

P_3, P_4 lok. minimum.

Pr 4

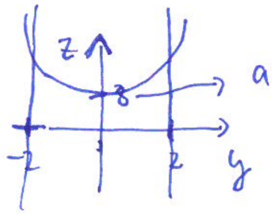
CVIKO

$f(x,y) = (2(x^2+y^2)+y^2)e^{-(x^2+y^2)} \leftarrow x^2+y^2=4$ dosadíme

$\tilde{f}(y) = (2\cdot 4 + y^2)e^{-4} = (8+y^2)\cdot e^{-4}$

$\wedge\ y \in [-2,2]$!

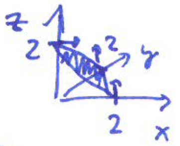
graf →



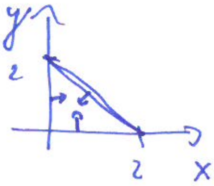
asi má lok. minimum v 0.

Z obrázku vidíme, že $\tilde{f}$ má lok. minimum v bode 0 t.j. f má lok. min v bode $[\overset{y}{0}, \overset{x}{\pm 2}]$ na M a $\tilde{f}$ má lok. maximum v bode ± 2 t.j. f má lok. maximum v bode $[\overset{y}{\pm 2}, \overset{x}{0}]$.

Pr5 $x=0,\ y=0,\ z=0,\ x+y+z=2$



rez $z=0:$



Pontivam $n\geq 1$

$$\int(-x+a)^n dx \,\Big|\, \begin{matrix} t=-x+a \\ dt=-dx \end{matrix} \Big| = \frac{-(-x+a)^{n+1}}{n+1}$$

Omko

$$0 \leq z \leq 2-(x+y)$$
$$0 \leq y \leq 2-x$$
$$0 \leq x \leq 2$$

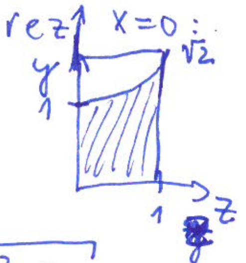
$$\int_0^2\int_0^{2-x}\int_0^{2-(x+y)} z^2\, dz\, dy\, dx =$$

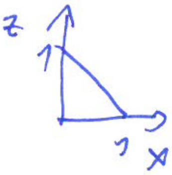
$$=\frac{1}{3}\int_0^2\int_0^{2-x}(2-(x+y))^3\, dy\, dx = \frac{1}{3}\int_0^2\left[-\frac{(2-(x+y))^4}{4}\right]_0^{2-x} dx$$

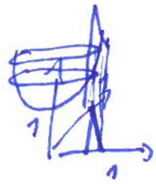
$$=\frac{1}{12}\int_0^2 (2-x)^4\, dx = \frac{1}{12}\left[-\frac{(2-x)^5}{5}\right]_0^2 = +\frac{1}{12}\,\frac{2^5}{5} = \frac{2^3}{15} = \frac{8}{15}$$

Pr6 $x^2+z^2+1=y^2,\ x+z=1,\ x,y,z\geq 0$

rez $x=0:$







$$0 \leq y \leq \sqrt{x^2+z^2+1}$$
$$0 \leq z \leq 1-x$$
$$0 \leq x \leq 1$$

$$\int_0^1\int_0^{1-x}\int_0^{\sqrt{x^2+z^2+1}} 2y\, dy\, dz\, dx =$$

$$=\int_0^1\int_0^{1-x} x^2+z^2+1\, dz\, dx = \int_0^1\left[(x^2+1)z+\frac{z^3}{3}\right]_0^{1-x} dx =$$

$$=\int_0^1 (x^2+1)(1-x)+\frac{(1-x)^3}{3}\, dx = \int_0^1 x^2+1-x^3-x+\frac{(1-x)^3}{3}\, dx$$

$$=\left[\frac{x^3}{3}+x-\frac{x^4}{4}-\frac{x^2}{2}-\frac{(1-x)^4}{4\cdot 3}\right]_0^1 = \frac{1}{3}+1-\frac{1}{4}-\frac{1}{2}+\frac{1}{12}$$

$$=\frac{4+12-3-6+1}{12}=\frac{8}{12}=\frac{2}{3}$$

P18 $x^2+y^2 \le a^2$, $a>0$

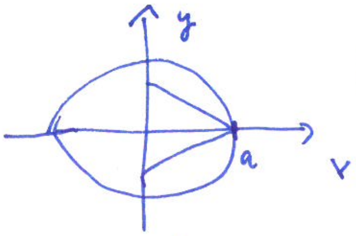
Transformácia F

$\tilde{x} = x - a$

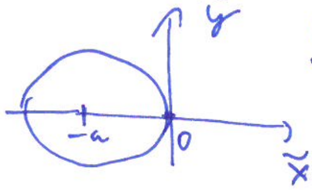
$\tilde{y} = y$

$\det F = 1$

$(\tilde{x}+a)^2 + \tilde{y}^2 \le a^2$



$\rho(x,y) = \sqrt{(x-a)^2+y^2}$



$\rho(\tilde{x},\tilde{y}) = \sqrt{\tilde{x}^2+\tilde{y}^2}$

Polárne súradnice:

$\tilde{x} = r\cdot\cos\varphi \qquad \varphi \in [\pi/2, -\pi/2]$

$\tilde{y} = r\cdot\sin\varphi \qquad r \in [0, -2a\cos\varphi]$

$(r\cos\varphi + a)^2 + r^2\sin^2\varphi \le a^2$

$r^2 + 2ra\cdot\cos\varphi \le 0$

$\underbrace{r}_{>0}(\underbrace{r+2a\cdot\cos\varphi}_{\le 0}) \le 0$

$r + 2a\cos\varphi \le 0$

$r \le -2a\cos\varphi$

$\rho(r,\varphi) = \sqrt{r^2} = r$

$$S = \int_{\frac{\pi}{2}}^{-\pi/2}\int_0^{-2a\cos\varphi} r\cdot r \, dr\, d\varphi = \int_{\frac{\pi}{2}}^{-\pi/2} -\frac{1}{3}\cdot 8a^3\cos^3\varphi \, d\varphi$$

$$= -\frac{8}{3}a^3 \int_{\frac{\pi}{2}}^{-\pi/2} \cos^3\varphi \, d\varphi \quad \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \end{array} \right|$$

$$= -\frac{8}{3}a^3 \int_1^{-1} (1-t^2)\, dt = \frac{8}{3}a^3 \int_{-1}^{1} 1-t^2 \, dt = \frac{8}{3}a^3\left[t - \frac{t^3}{3}\right]_{-1}^{1}$$

$$= \frac{8}{3}a^3\cdot 2\cdot\left(1-\frac{1}{3}\right) = \frac{32}{9}a^3$$

$y_T = 0$ zrejme zo symetrie oblasti

$$x_T = \frac{1}{S}\int_{\pi/2}^{-\pi/2}\int_0^{-2a\cos\varphi} r^2\cdot(a + r\cdot\cos\varphi)\, dr\, d\varphi$$

$$= \frac{1}{S}\left[\underbrace{\int_{\pi/2}^{-\pi/2}\int_0^{-2a\cos\varphi} r^2\cdot a\, dr\, d\varphi}_{a\cdot S} + \int_{\pi/2}^{-\pi/2}\int_0^{-2a\cos\varphi} r^3\cos\varphi\, dr\, d\varphi\right] =$$

$$= a + \frac{1}{S}\underbrace{\int_{\pi/2}^{-\pi/2}\int_0^{-2a\cos\varphi} r^3\cos\varphi\, dr\, d\varphi}_{(*)} = a - \frac{64}{15}a^4\cdot\frac{9}{32 a^3} = a - \frac{6}{5}a = -\frac{1}{5}a$$

$$(*) = \int_{\pi/2}^{-\pi/2} \frac{1}{4}(2a\cos\varphi)^4\cos\varphi\, d\varphi = \frac{16}{4}a^4 \int_{\pi/2}^{-\pi/2}\cos^5\varphi\, d\varphi \quad \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \end{array} \right|$$

$$= \frac{16}{4}a^4 \int_1^{-1}(1-t^2)^2\, dt = \frac{16}{4}a^4\left[t - 2\frac{t^3}{3} + \frac{t^5}{5}\right]_1^{-1} = -\frac{16}{4}a^4\cdot 2\cdot\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= a^4 - 8\,\frac{15-10+3}{15} = -\frac{64}{15}a^4$$