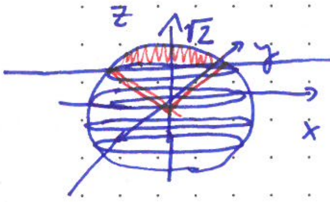
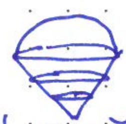
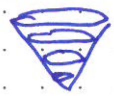
(1) $x^2+y^2+z^2=2$, $z=1$



Budeme počítať:

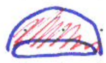
A. objem „zmrzliny" B. objem kužela (kornútka)



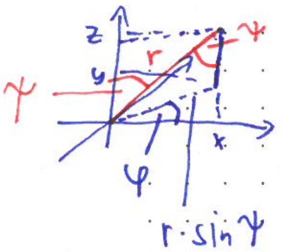


GUĽOVÝ VÝSEK

a rozdiel bude náš božský kopček



Výpočet A. (sférické súradnice)



$x = r\cdot\cos\varphi\cdot\sin\psi$

$y = r\cdot\sin\varphi\cdot\sin\psi$

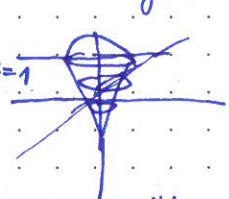
$z = r\cdot\cos\psi$

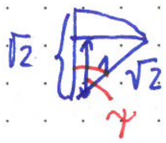
$|J| = r^2\sin\psi$

$\psi\in[0,\pi)$

$\varphi\in[0,2\pi)$

A. Potrebujeme určiť r, φ, ψ.





$\cos\psi = \frac{1}{\sqrt2} \Rightarrow \psi = \frac{\pi}{4}$

$\varphi\in[0,2\pi]$

$r\in[0,\sqrt2]$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt2} r^2\sin\psi\,dr\,d\psi\,d\varphi =$$

$$= \frac{1}{3}\cdot[r^3]_0^{\sqrt2}\,[-\cos\psi]_0^{\pi/4}\cdot 2\pi$$

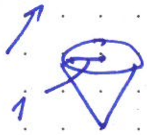
$$= \frac{2}{3}\sqrt2\left(-\frac{1}{\sqrt2}+1\right)\cdot 2\pi$$

$$= \frac{4\pi}{3}(\sqrt2-1)$$

B. Objem kornútka – stredná škola

$\frac{1}{3} S_p\cdot v = \frac{1}{3}\cdot\pi(1)^2\cdot 1 = \frac{\pi}{3}$

Podstava $z=1$



alebo:

$x = r\cdot\cos\varphi$ $\quad r\in[0,z]$

$y = r\cdot\sin\varphi$ $\quad \varphi\in[0,2\pi]$

$z = z$

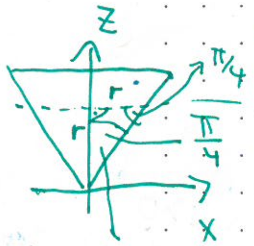
$|J| = r$

$$\int_0^1\int_0^z\int_0^{2\pi} r\,d\varphi\,dr\,dz$$

VALCOVÉ SÚRADNICE

$$= 2\pi\int_0^1\left[\frac{r^2}{2}\right]_0^z dz$$

$$= \pi\int_0^1 z^2\,dz = \pi\left[\frac{z^3}{3}\right]_0^1 = \frac{\pi}{3}$$



rovnoramenný △

(C) rozdiel

$\frac{4\pi}{3}(\sqrt2-1)-\frac{\pi}{3}$ je výsledok.

② $(1+e^x)y\dfrac{dy}{dx} = e^x$

$$\int y\,dy = \int \frac{e^x}{\underset{\neq 0\ \text{ok}}{1+e^x}}\,dx \;\left|\begin{array}{l} u = 1+e^x \\ du = e^x dx \end{array}\right| = \int \frac{1}{u}\,du = \ln|u| = \ln|1+e^x|$$

$$\frac{y^2}{2} = \underbrace{\ln|1+e^x|}_{\text{kladné}} + c \quad , \; c \in \mathbb{R} \text{ vhodné}$$

$$y^2 = 2\ln|1+e^x| + c$$

$$y = \sqrt{2\ln|1+e^x| + c}$$

(keď mák čas, vždy vyrobte skúšku správnosti!)

Počiatočná podmienka $y(0)=1$ $\quad 1 = y(0) = \sqrt{2\ln 1 + c} = \sqrt{c} \Rightarrow c = 1$

$$y = \sqrt{2\ln|1+e^x| + 1}$$

③ $\dfrac{dy}{dx} = y' = \dfrac{y^2+1}{x+1}$

$$\int \frac{1}{\underset{\neq 0}{y^2+1}}\,dy = \int \frac{1}{x+1}\,dx$$

$\to x \neq 1$ predpoklad

$$\operatorname{arctg}(y) \;\; = \ln|x+1| + c \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \quad , c \in \mathbb{R}$$

$$y = tg(\ln|x+1| + c).$$

④ $xy' + y\ln(x) = y\ln(y) \Rightarrow x, y > 0$!

$$xy' = y(\ln(y) - \ln(x)) = y\ln\left(\frac{y}{x}\right)$$

$$y' = \frac{y}{x}\ln\left(\frac{y}{x}\right) \qquad \text{substitúcia: } \boxed{u = \frac{y}{x}}$$

$$u'(x) = \frac{y'\cdot x - y}{x^2} = \frac{y'}{x} - \frac{y}{x^2} = \frac{y'}{x} - \frac{u}{x}$$

$$xu'(x) + u = u\ln(u)$$

$$xu'(x) = u(\ln(u) - 1) \qquad u'(x) + \frac{u}{x} = \frac{y'}{x} \Rightarrow \boxed{y' = xu'(x) + u}$$

$$\int \frac{1}{u(\ln(u)-1)}\,du = \int \frac{1}{x}\,dx$$

$u \neq 0$ ok $\quad$ predpoklad $\ln(u) - 1 \neq 0$ t.j. $\ln(u) = 1$ musíme overiť zvlášť!

$$\left|\begin{array}{l} \ln|u| - 1 = t \\ \frac{1}{u}du = dt \end{array}\right| \quad = \ln|x| + c \quad , c \in \mathbb{R}$$

$$\int \frac{1}{t}\,dt = \ln|t| = \boxed{\ln|\ln|u| - 1| = \ln|x| + c}$$

$e^c =: K > 0$

$$|\ln|u| - 1| = e^{\ln|x| + c} = |x|\cdot e^c = |x|\cdot K$$

$:= \pm K$, $\neq 0$ $\qquad u > 0$

$$\ln|u| = \pm|x|\cdot K + 1 \Rightarrow |u| = e^{K|x|+1} \Rightarrow \boxed{u = \pm e^{K|x|+1}}$$

$$\frac{y}{x} = \pm e^{L|x|+1} \Rightarrow y = \pm x\, e^{L|x|+1} = \underline{\underline{x \cdot e^{L\cdot x+1}}}, \quad L \neq 0$$
$y>0$ $\quad x>0$

Ešte treba doriešiť

$$\ln|u| = 1 \Rightarrow u = e \Rightarrow \frac{y}{x} = e \Rightarrow y = e \cdot x$$
$u>0$

$$\boxed{y = x \cdot e^{L\cdot x+1}, \quad L \in \mathbb{R}}$$

homogénna D.R.

Definícia homogénnej D.R.: $P(x,y)\,dx + Q(x,y)\,dy = 0$ je homogénna D.R., ak $P(x,y)$, $Q(x,y)$ sú homogénne polynómy rovnakého stupňa k

t.j. $P(tx,ty) = t^k P(x,y)$ a $Q(tx,ty) = t^k Q(x,y)$

③ Lineárna d.r. 1. rádu $\underline{\underline{y' = a(x)y + b(x)}}$

$$y' = x - \frac{2y}{x^2-1}$$

1. riešime homogénnu t.j. $b(x)=0$

$$y' = \frac{-2y}{x^2-1} \rightarrow x \neq \pm 1$$

$$-\frac{1}{2y}\,dy = \frac{1}{x^2-1}\,dx$$

budeme musieť overiť, či $y=0$ nie je riešenie!

$$-\frac{1}{2}\ln|y| = \int \frac{1/2}{x-1} - \frac{1/2}{x+1}\,dx = \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + c = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \frac{c}{2}$$

$$\ln|y| = -\ln\left|\frac{x-1}{x+1}\right| + c = \ln\left|\frac{x-1}{x+1}\right|^{-1} + c = \ln\left|\frac{x+1}{x-1}\right| + c$$

$$|y| = e^c \cdot \left|\frac{x+1}{x-1}\right| = k \cdot \left|\frac{x+1}{x-1}\right|, \quad k := e^c > 0$$

$$y = \pm k \left|\frac{x+1}{x-1}\right| = L \cdot \frac{x+1}{x-1} \qquad L \neq 0$$

(vždy rieši Lin. D.R.)

Overíme, či $y=0$ je riešenie homogénnej D.R. $0=0$ ✓ platí t.j.

$$\boxed{y = L \cdot \frac{x+1}{x-1}, \quad L \in \mathbb{R}}$$

VARIÁCIA KONŠTANTY:

$$y = L(x)\cdot\frac{x+1}{x-1} \Rightarrow y' = L'(x)\cdot\frac{x+1}{x-1} + L(x)\frac{x-1-(x+1)}{(x-1)^2} = L'(x)\frac{x+1}{x-1} + L(x)\frac{(-2)}{(x-1)^2}$$

$$L'(x)\frac{x+1}{x-1} - \frac{2L(x)}{\cancel{(x-1)^2}} = x - \frac{2L(x)\frac{x+1}{x-1}}{x^2-1} = x - \frac{2L(x)}{\cancel{(x-1)^2}}$$

musí vždy vypadnúť!

$$L'(x)\frac{x+1}{x-1} = x \qquad\qquad x-1 = (x+1)-2$$

$$\int L'(x)\,dx = \int \frac{x(x-1)}{x+1}\,dx \Rightarrow L(x) = \int \frac{x\cdot[(x+1)-2]}{x+1}\,dx$$

$$x = (x+1)-1$$

$$L(x) = \int x - \frac{2x}{x+1}\,dx = \int x - 2 + \frac{2}{x+1}\,dx = \boxed{\frac{x^2}{2} - 2x + 2\ln|x+1| + c}$$

celkovo

$$y = \frac{x+1}{x-1}\cdot\left(\frac{x^2}{2} - 2x + 2\ln|x+1| + c\right),\ \underline{c\in\mathbb{R}}$$

Integračný faktor:

$$(a(x)\cdot b(x))' = a'(x)\,b(x) + a(x)\cdot b'(x)$$

$$y' + \frac{2y}{x^2-1} = x \quad \Big|\ e^{\int\frac{2}{x^2\pm1}dx} \longrightarrow \boxed{\text{Integračný faktor}}$$

$$y'\cdot e^{\int\frac{2}{x^2\pm1}dx} + \frac{2}{x^2\pm1}\cdot e^{\int\frac{2}{x^2\pm1}}\cdot y = x\, e^{\int\frac{2}{x^2\pm1}dx}$$

$$\left(y\cdot e^{\int\frac{2}{x^2\pm1}dx}\right)' = x\, e^{\int\frac{2}{x^2\pm1}dx}$$

$$y\cdot e^{\int\frac{2}{x^2\pm1}dx} = \int x\cdot e^{\int\frac{2}{x^2\pm1}}dx$$

$$\int\frac{2}{x^2\pm1}dx = 2\int\frac{1}{x^2-1}dx = 2\left(\int\frac{-1/2}{x+1} + \frac{1/2}{x-1}\,dx\right) = \ln\left|\frac{x\mp1}{x+1}\right|$$

$$e^{\ln\left|\frac{x\mp1}{x+1}\right|} = \left|\frac{x\mp1}{x+1}\right|$$ (môžete zobrať aj ľubovoľný nenulový násobok, preto nemusíte ukázať |.|, aj každú zobrať ľubovoľnú primitívnu funkciu!

$$y = \frac{x+1}{x\pm1}\cdot\left(\int x\left(\frac{x\mp1}{x+1}\right)dx\right) = \frac{x+1}{x\mp1}\left(\frac{x^2}{2} - 2x + 2\ln|x+1| + c\right)$$

$y(0)=-1$, potom $-1 = -1c \Rightarrow c = 1$

$$y = \frac{x-1}{x+1}\left(\frac{x^2}{2} - 2x + 2\ln|x-1| + 1\right)$$

$3 = y(2) = \frac{1}{3}(2 - 4 + 2\ln|3| + c)$ (výpočet vyššie) $\Rightarrow 11 = 2\ln|3| + c$

$\underline{c = 11 - 2\ln 3}$