Filters in Image Processing

Fourier Transform in one dimension

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Outline

Fourier Transform

Jean Baptiste Joseph Fourier (1768–1830)

Fourier Transform

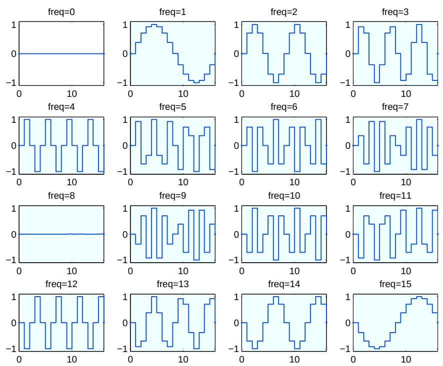
Basis functions

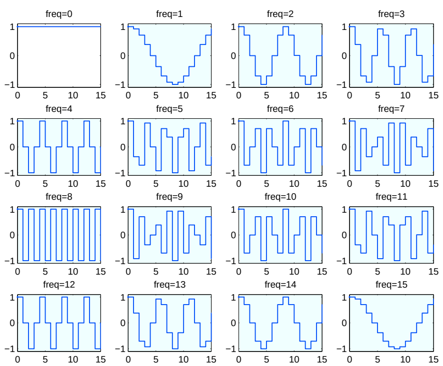
Common request:

- the basis should be orthonormal, i.e. $\varphi_k \cdot \varphi_l = \delta_{k,l}$

$$\varphi_k(m) = \frac{1}{\sqrt{N}} e^{\frac{2\pi imk}{N}} = \frac{1}{\sqrt{N}}\left(\cos\frac{2\pi mk}{N} + i\sin\frac{2\pi mk}{N}\right)$$

- the basis functions for $N = 16$:





Fourier Transform

Basis functions

Properties

- periodical:

$$\varphi_{k+N}(m) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m(k+N)}{N}} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m k}{N}} = \varphi_k(m)$$

- symmetrical:

$$\begin{aligned}\varphi_{N-k}(m) &= \frac{1}{\sqrt{N}} e^{\frac{2\pi i m(N-k)}{N}} \\ &= \frac{1}{\sqrt{N}} e^{-\frac{2\pi i m k}{N}} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i(-m)k}{N}} = \varphi_k(-m)\end{aligned}$$

- orthonormal:

$$\varphi_k(m) \cdot \varphi_l(m) = \sum_{m=0}^{N-1} \frac{1}{\sqrt{N}} e^{\frac{2\pi m k i}{N}} \overline{\frac{1}{\sqrt{N}} e^{\frac{2\pi m l i}{N}}} = \frac{1}{N} \sum_{m=0}^{N-1} e^{\frac{2\pi m(k-l)i}{N}} = \delta_{k,l}$$

Fourier Transform

Definition

Given 1D discrete function f of N samples and a basis $(\varphi_k(m), k = \{0, \dots, N-1\})$, let us define:

- forward 1D discrete Fourier transform:

$$\mathcal{F}(k) \equiv f \cdot \varphi_k = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}}$$

- inverse 1D discrete Fourier transform:

$$f(m) \equiv \mathcal{F} \cdot \overline{\varphi_m} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathcal{F}(k) e^{\frac{2\pi i m k}{N}}$$

$$f \cdot \varphi_k = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) \overline{e^{\frac{2\pi i m k}{N}}} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}}$$

Fourier Transform

Matrix notation

If

$$\overline{\varphi_k(m)} = \frac{1}{\sqrt{N}} e^{\frac{-2\pi i m k}{N}} = \frac{1}{\sqrt{N}} \left(e^{-\frac{2\pi i}{N}}\right)^{mk} = \frac{1}{\sqrt{N}} \psi^{mk}$$

then

$$A = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \psi^1 & \psi^2 & \dots & \psi^{N-1} \\ 1 & \psi^2 & \psi^{2\cdot 2} & \dots & \psi^{(N-1)2} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi^{N-1} & \psi^{2(N-1)} & \dots & \psi^{(N-1)(N-1)} \end{bmatrix}$$

and

$$\mathcal{F} = Af \quad \Rightarrow \quad f = A^{-1}\mathcal{F} = \overline{A}^T \mathcal{F}$$

Notice: $\psi = e^{-\frac{2\pi i}{N}}$ is called the N^{th} root of unity.

Fourier Transform

The obvious question

When we apply FT, we usually say "let us decompose our signal into the sine waves ..." Why do we use another (so complicated) basis?

Basis function is a "sine wave"

- we avoid complex numbers
- more intuitive if basis function is a simple "sine wave"
- sine waves without phase shift do not generate the whole space
- possible basis function: $\sin(km - \alpha)$
- α hidden in the sine function spoils the linearity; matrix multiplication cannot be used

Basis function is $\varphi_k(m)$

- we have to use complex numbers
- $\varphi_k(m)$ functions generate the whole space (form basis)
- this basis is orthonormal
- transform is linear, i.e. realized via matrix multiplication

Fourier Transform

The meaning of Fourier coefficients

If you perform inverse FT

$$f(m) = \mathcal{F} \cdot \overline{\varphi_m} = \sum_{k=0}^{N-1} \mathcal{F}(k)\overline{\overline{\varphi_k(m)}} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathcal{F}(k) e^{i\frac{2\pi km}{N}}$$

you literally compose the original signal f by combining the individual basis functions.

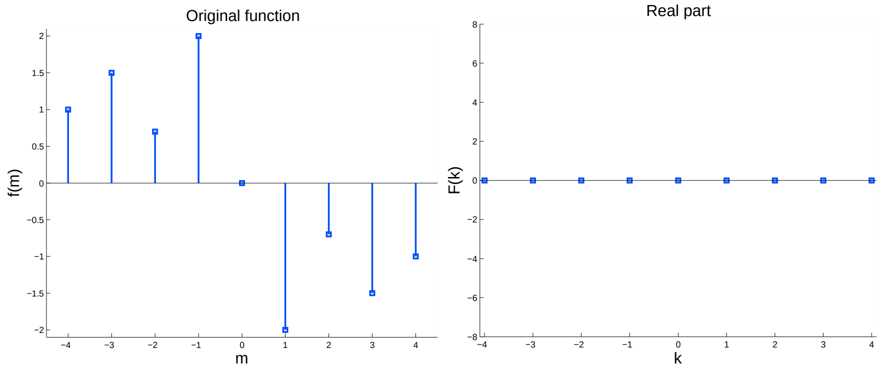
- Each basis function $\varphi_k(m) = e^{i\frac{2\pi mk}{N}}$ defines only the frequency.
- Fourier coefficient $\mathcal{F}(k) = |z_k|\,(\cos\alpha_k + i\sin\alpha_k) = |z_k|e^{i\alpha_k}$ modifies the corresponding basis function $\varphi_k(m)$ by scaling it and shifting it.

$$\begin{aligned} \mathcal{F}(k)\overline{\overline{\varphi_m(k)}} &= |z_k|e^{i\alpha_k}e^{i\frac{2\pi mk}{N}} = |z_k|e^{i\alpha_k + i\frac{2\pi mk}{N}} \\ &= |z_k|\left\{\cos\left(\frac{2\pi k}{N}m + \alpha_k\right) + i\sin\left(\frac{2\pi k}{N}m + \alpha_k\right)\right\} \end{aligned}$$

Fourier Transform

The meaning of Fourier coefficients

The number of samples is equal, i.e. $|f| = |\mathcal{F}| = |\varphi_k|$



Fourier Transform

The meaning of Fourier coefficients

- $\mathcal{F}(0)$ matches the lowest frequency in the signal f and corresponds to the "mean" of f:

$$\mathcal{F}(0) \equiv \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)e^{-\frac{2\pi i m0}{N}} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)$$

- $\mathcal{F}(0)$ is usually called DC term
  DC . . . "direct current" (current of zero frequency)
- $\mathcal{F}(1)$ . . . $\mathcal{F}(N-1)$ are called AC terms
  AC . . . "alternating current"
- $\mathcal{F}(\frac{N-1}{2})$ matches the highest frequency in the signal f

Exercise: Why does the component $\mathcal{F}(\frac{N-1}{2})$ correspond to the highest frequency?

Fourier Transform

Continuous version

We again deal with a projection

- the basis functions are:

$$\varphi_\omega(x) = e^{2\pi i x\omega} = \cos 2\pi x\omega + i\sin 2\pi x\omega$$

- the projection of the function f onto the basis function $\varphi_\omega(x)$ is the inner product as well as in the discrete case:

$$f \cdot \varphi_\omega = \int_a^b f(x)\overline{\varphi_\omega(x)}dx$$

Fourier Transform

Definition

Given 1D integrable function f and a basis $(\varphi_\omega, \omega \in \mathbb{R})$, let us define:

- forward 1D continuous Fourier transform

$$\mathcal{F}(\omega) \equiv \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx$$

- inverse 1D continuous Fourier transform

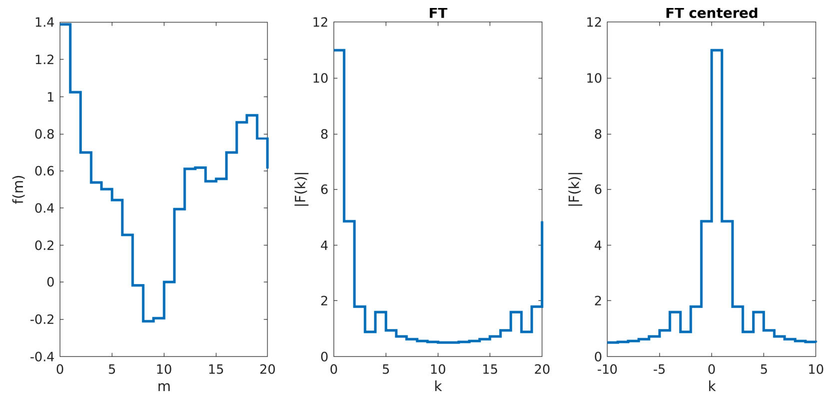
$$f(x) \equiv \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{2\pi i x \omega} d\omega$$

Notice: The sign flip explanation is beyond the scope of this course.

Fourier spectrum visualization

Centering

Analysis of zero frequency

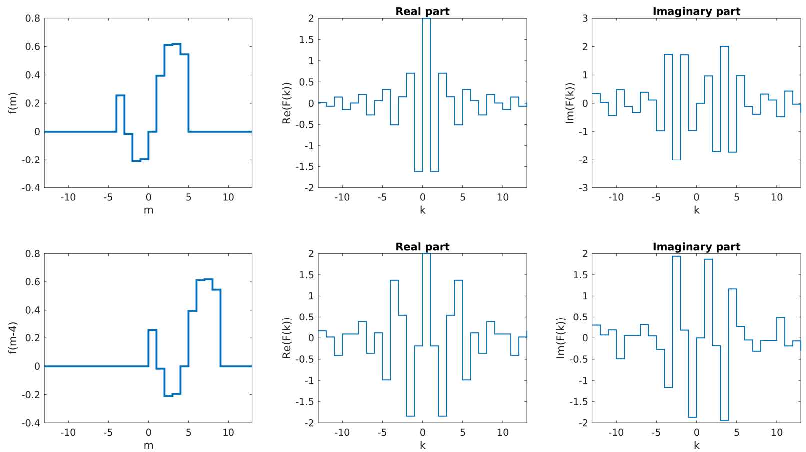


Notice: Verify in FTutor1D (`https://cbia.fi.muni.cz/software/ftutor1d.html`).

Fourier spectrum visualization

Real and Imaginary part

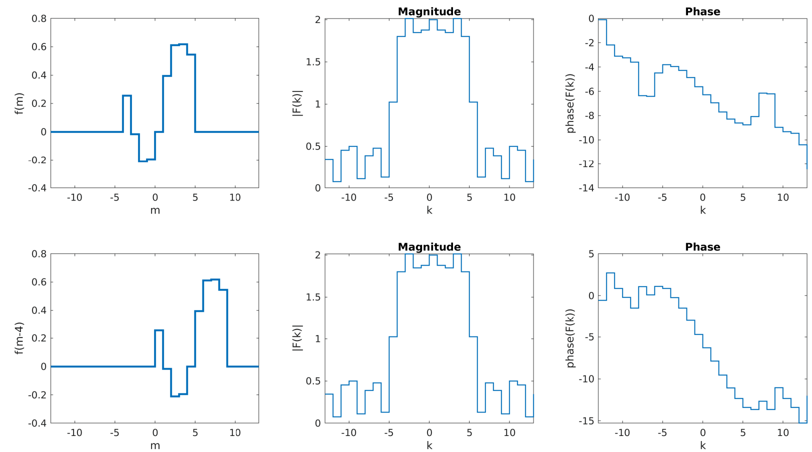
Analysis of time shift



Fourier spectrum visualization

Magnitude and phase

Analysis of time shift



FT – Properties & Theorems

Oddness and evenness

1. Each function $f(x)$ is sum of its odd and even part:

$$
\begin{aligned}
E(x) &= \frac{1}{2}[f(x) + f(-x)] \\
O(x) &= \frac{1}{2}[f(x) - f(-x)] \\
f(x) &= E(x) + O(x)
\end{aligned}
$$

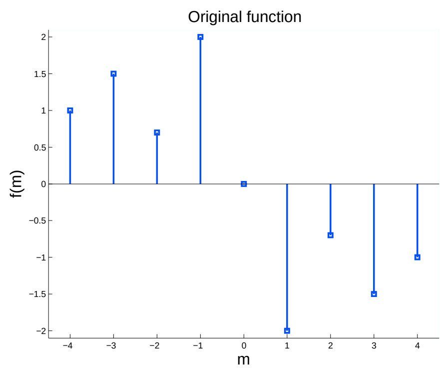
2. $\sin(x)$ is an odd function
   $\cos(x)$ is an even function
3. Any FT basis function is composed of sine and cosine waves

Corollary:

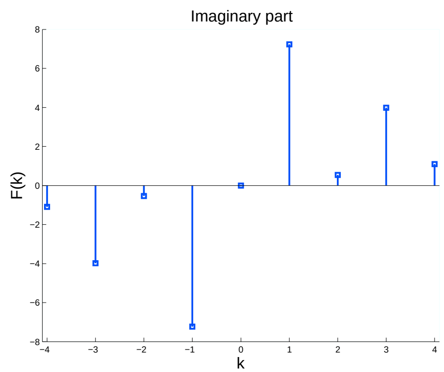
- FT of even function misses imaginary part (sine waves)
- FT of odd function misses real part (cosine waves)

FT – Properties & Theorems

Oddness



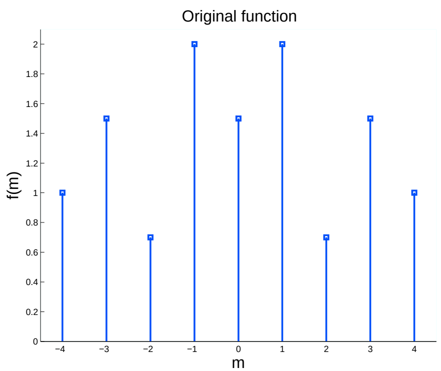
original function (odd)



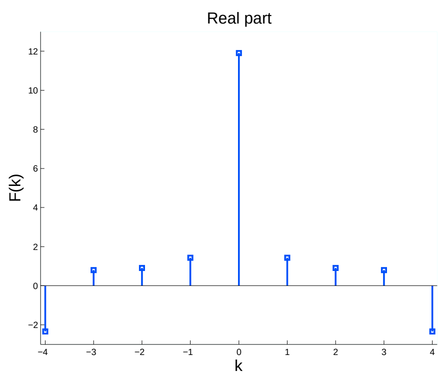
its transform pair

FT – Properties & Theorems

Evenness



original function (even)



its transform pair

FT – Properties & Theorems

Transform pairs

original function	transform pair
even	even
odd	odd
real and even	real and even
real and odd	imaginary and odd
real	complex and hermitian

Hermitian ... conjugated symmetry, i.e. $\mathcal{F}(-x) = \overline{\mathcal{F}(x)}$

Exercise: How does change the FT if the original function is "real and even"? Prove your assertion.

FT – Properties & Theorems

Transform pairs

If $\mathcal{F}(\omega) = FT\left[f(x)\right](\omega)$ we usually write:

$$f(x) \supset \mathcal{F}(\omega)$$

and call $f(x)$ and $\mathcal{F}(\omega)$ the Fourier transform pair.

Keep in mind that:

- not all the continuous functions are integrable, i.e. suitable for FT. For example $\sin x, \cos x, f(x) = 1$,
- any discrete function is suitable for FT, since it is finite,
- if f is even then $|\mathcal{F}(\omega)| = |Re(\mathcal{F}(\omega))|$, i.e. imaginary part is null,
- it is common to say that $f(x)$ belongs to time domain.

FT – Properties & Theorems

Transform pairs

Statement:

Gaussian $\supset$ Gaussian

Proof:

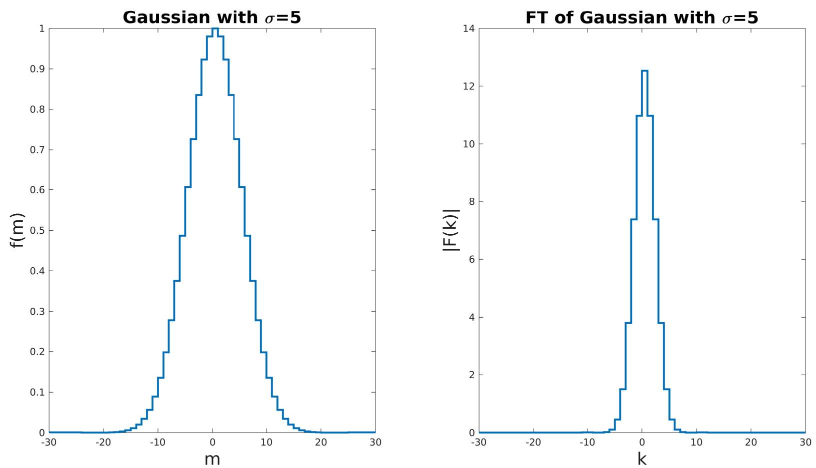
$$\begin{aligned} FT\left[e^{-ax^2}\right](\omega) &= \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i x\omega}\, dx \\ &= \int_{-\infty}^{\infty} e^{-ax^2} \left(\cos 2\pi x\omega - i \sin 2\pi x\omega\right) dx \\ &\vdots \\ &= \sqrt{\frac{\pi}{a}}\, e^{-\left(\frac{\pi^2}{a}\right)\omega^2} \end{aligned}$$

$\square$

FT – Properties & Theorems

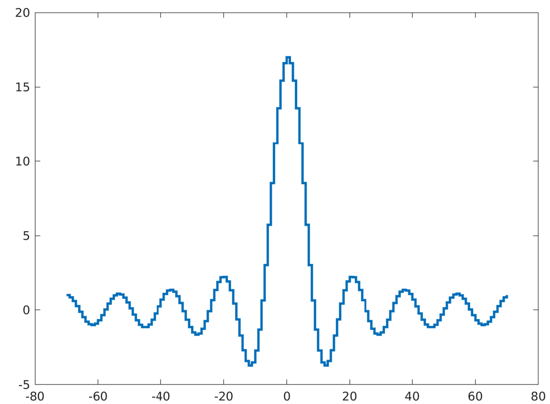
Transform pairs

Gaussian $\supset$ Gaussian



FT – Properties & Theorems

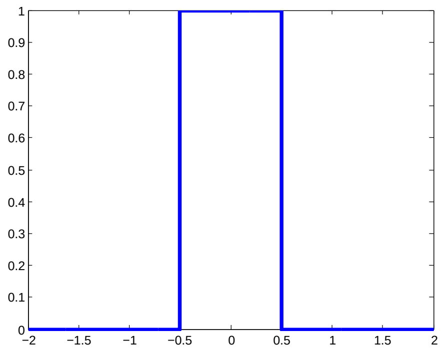
"sinc" function and its importance



$$\mathrm{sinc}(x) = \frac{\sin x}{x}$$

FT – Properties & Theorems

"Π" function and its importance



$$\Pi(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ (\frac{1}{2} & |x| = \frac{1}{2}) \\ 0 & |x| > \frac{1}{2} \end{cases}$$

FT – Properties & Theorems

Transform pairs

Statement:

Rectangle Π function ⊃ Sinc

Proof:

$$FT\left[\Pi(ax)\right](\omega) = \int_{-\infty}^{\infty} \Pi(ax) e^{-2\pi i x\omega} dx$$

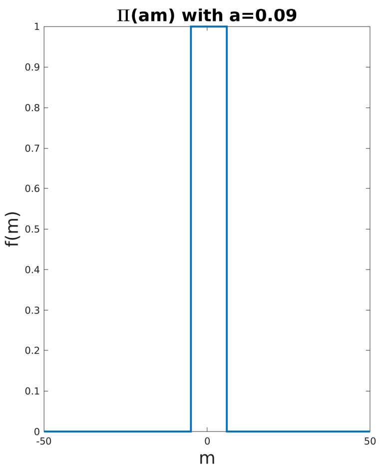
$$= \int_{-\frac{1}{2a}}^{\frac{1}{2a}} 1 \cdot (\cos 2\pi x\omega - i \sin 2\pi x\omega)\, dx = \cdots = \frac{1}{a} sinc\left(\frac{\pi}{a}\omega\right)$$

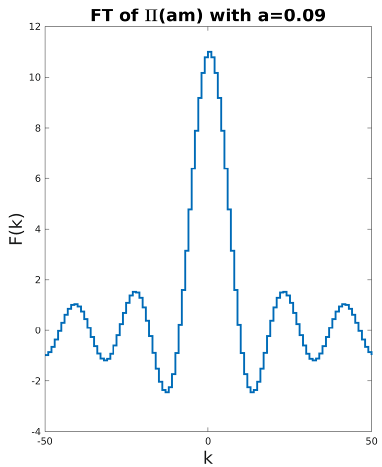
□

FT – Properties & Theorems

Transform pairs

Rectangle Π function ⊃ Sinc





FT theorems

Scaling

Statement:

$$f(ax) \supset \frac{1}{|a|}\mathcal{F}\left(\frac{\omega}{a}\right)$$

Proof:

$$FT\left[f(ax)\right](\omega) = \int_{-\infty}^{\infty} f(ax) e^{-2\pi i x\omega} dx$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} f(ax) e^{-2\pi i (ax)(\omega/a)} d(ax)$$

$$= \frac{1}{|a|}\mathcal{F}\left(\frac{\omega}{a}\right)$$

□

Notice: Stretch in time domain corresponds to shrinkage in Fourier domain and vice versa.

FT theorems

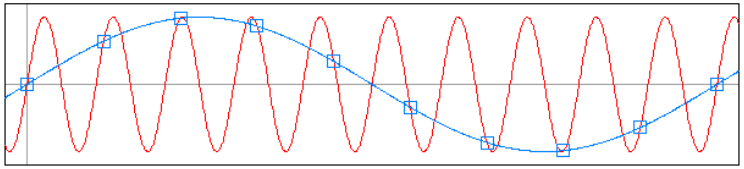
Scaling

Obstacles in discrete domain

- Continuous domain: scaling
- Discrete domain: upsampling/downsampling (loss of information)

$$f(am) \approx \text{downsample}_{a-\text{times}}(f(m))$$

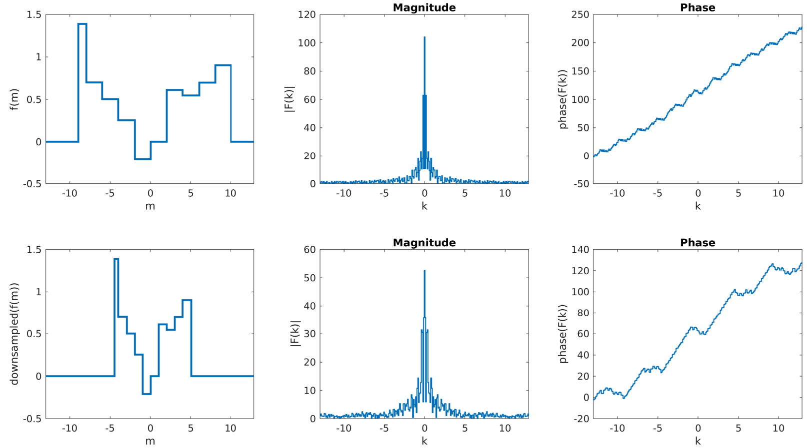
- Sampling may introduce new frequencies



FT theorems

Scaling

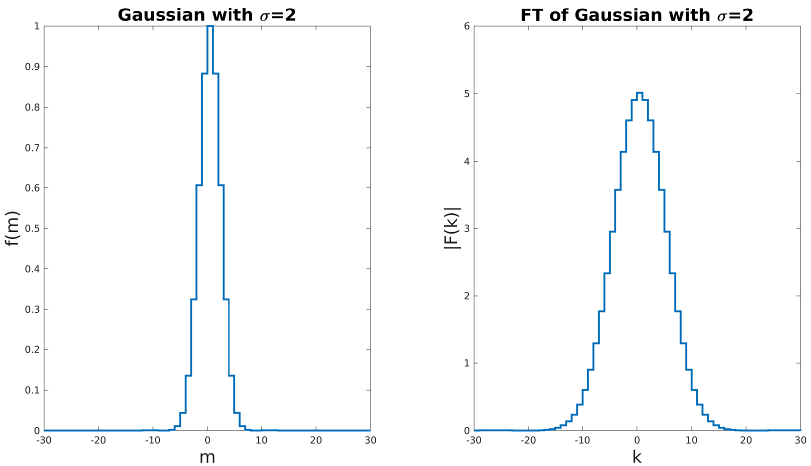
Fine sampling



Rough sampling

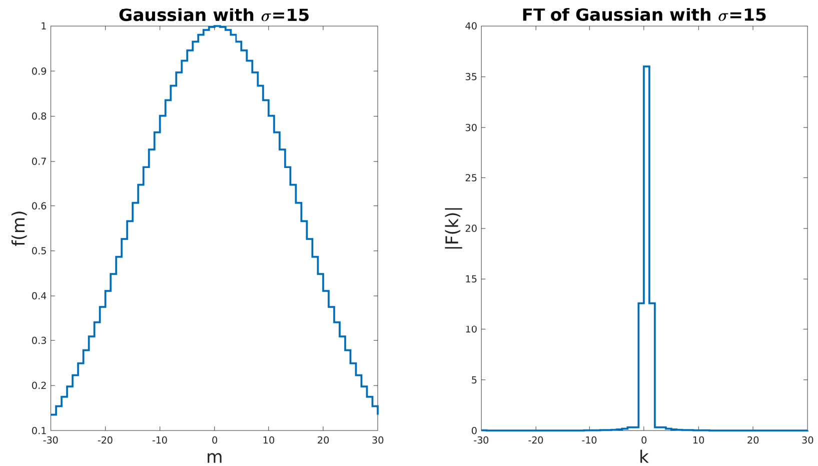
FT theorems

Scaling/Reciprocity



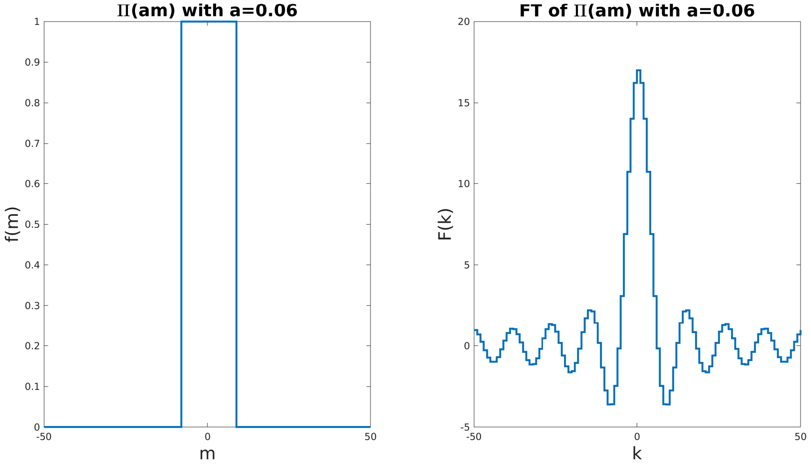
FT theorems

Scaling/Reciprocity



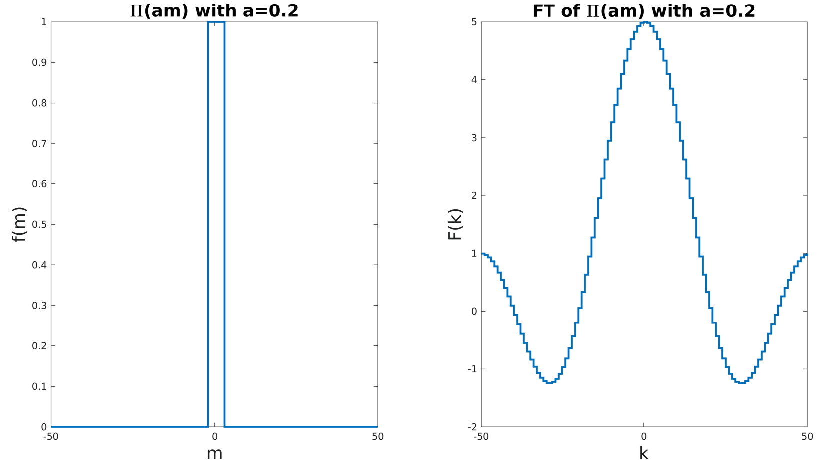
FT theorems

Scaling/Reciprocity



FT theorems

Scaling/Reciprocity



FT theorems

Shift

Statement:

$$f(x-a) \supset e^{-2\pi i a\omega}\mathcal{F}(\omega)$$

Proof:

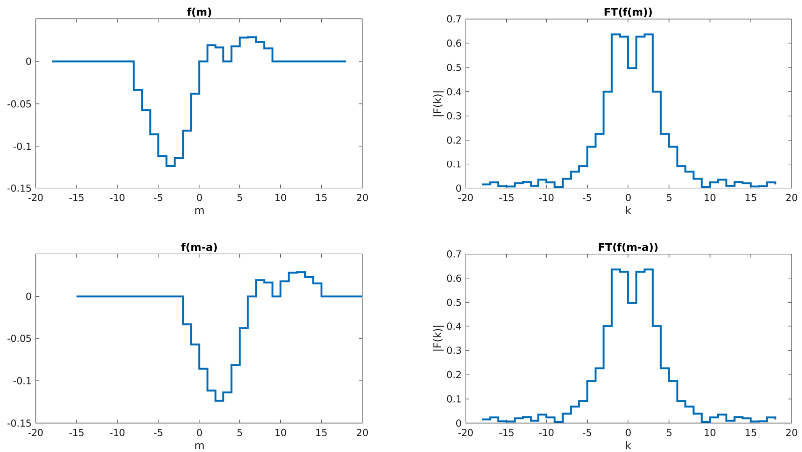
$$\begin{aligned} FT\left[f(x-a)\right](\omega) &= \int_{-\infty}^{\infty} f(x-a)e^{-2\pi i x\omega}dx \\ &= \int_{-\infty}^{\infty} f(x-a)e^{-2\pi i(x-a)\omega}e^{-2\pi i a\omega}d(x-a) \\ &= e^{-2\pi i a\omega}\mathcal{F}(\omega) \end{aligned}$$

□

Notice: Shift affects only phase. The higher the frequency ω is the more the corresponding cosine wave is affected.

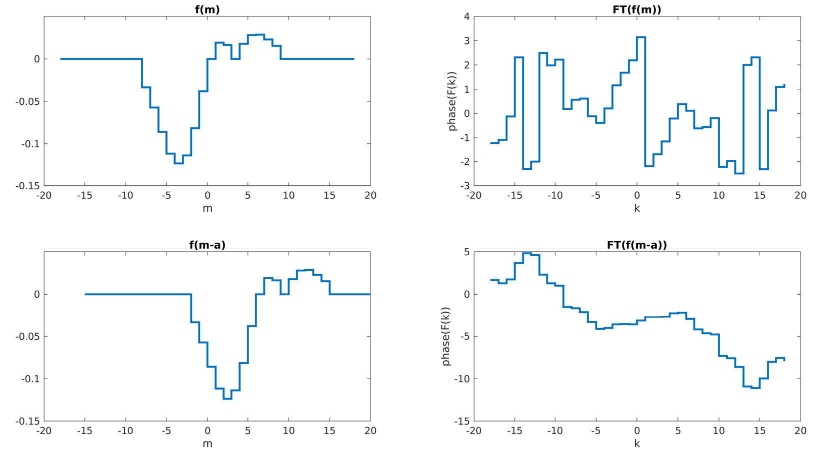
FT theorems

Shift



FT theorems

Shift



FT theorems

Linearity

Statement:

$$\alpha f(x) + \beta g(x) \supset \alpha \mathcal{F}(\omega) + \beta \mathcal{G}(\omega)$$

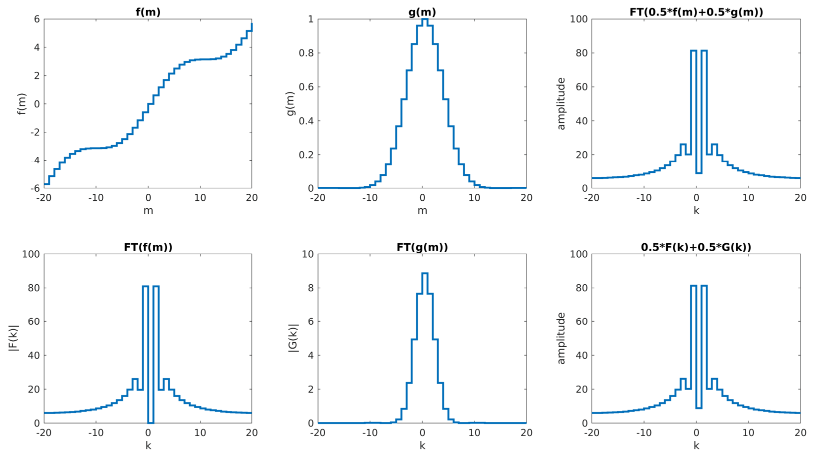
Proof:

$$\begin{aligned} FT\left[\alpha f(x) + \beta g(x)\right](\omega) &= \int_{-\infty}^{\infty} \left[\alpha f(x) + \beta g(x)\right] e^{-2\pi i x \omega} dx \\ &= \alpha \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx + \beta \int_{-\infty}^{\infty} g(x) e^{-2\pi i x \omega} dx \\ &= \alpha \mathcal{F}(\omega) + \beta \mathcal{G}(\omega) \end{aligned}$$

□

FT theorems

Linearity



FT theorems

Convolution theorem

Statement:

$$\begin{aligned} f(x) * g(x) &\supset \mathcal{F}(\omega)\mathcal{G}(\omega) \\ f(x)g(x) &\supset \mathcal{F}(\omega) * \mathcal{G}(\omega) \end{aligned}$$

Proof:

$$\begin{aligned} FT\left[f(x) * g(x)\right](\omega) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')g(x-x')dx' \right] e^{-2\pi i x \omega} dx \\ &= \int_{-\infty}^{\infty} f(x') \left[\int_{-\infty}^{\infty} g(x-x') e^{-2\pi i x \omega} dx \right] dx' \\ &= \int_{-\infty}^{\infty} f(x') e^{-2\pi i x' \omega} \mathcal{G}(\omega) dx' = \mathcal{F}(\omega)\mathcal{G}(\omega) \end{aligned}$$

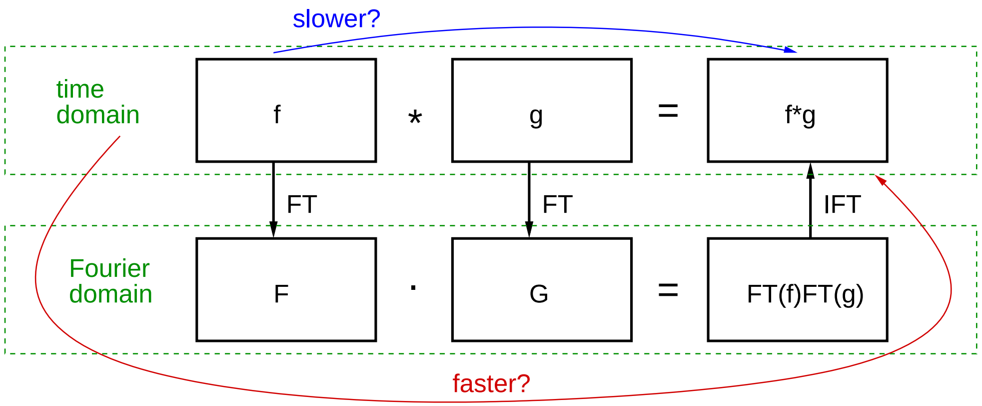
□

FT theorems

Convolution theorem

The meaning:

- The convolution in time domain corresponds to point-wise multiplication in the Fourier domain, and vice versa.



FT theorems

Rayleigh's energy theorem (Parseval's theorem)

Statement:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\mathcal{F}(\omega)|^2 d\omega$$

Proof:

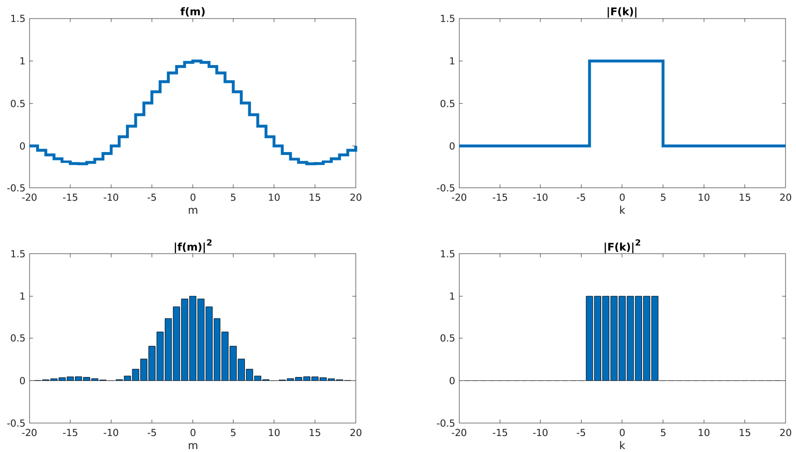
$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx \\
&= \int_{-\infty}^{\infty} f(x)\overline{f(x)}e^{-2\pi i x\omega'}dx & \omega' = 0 \\
&= \mathcal{F}(\omega') * \overline{\mathcal{F}(-\omega')} & \omega' = 0 \\
&= \int_{-\infty}^{\infty} \mathcal{F}(\omega')\overline{\mathcal{F}(\omega-\omega')}d\omega & \omega' = 0 \\
&= \int_{-\infty}^{\infty} \mathcal{F}(\omega)\overline{\mathcal{F}(\omega)}d\omega
\end{aligned}$$

□

FT theorems

Rayleigh's energy theorem (Parseval's theorem)

Rayleigh's energy theorem – the integral of the square of a function is equal to the integral of the square of its transform.



Notice: $|\mathcal{F}(k)|^2$ is called a power spectrum.

DFT theorems

Convolution theorem

Let f and g be 1D discrete periodic signals of length N, then:

$$f * g \;=\; IDFT\left[DFT(f) \cdot DFT(g)\right]\sqrt{N}$$

Proof:

$$\begin{aligned}
f(m) * g(m) &= \sum_{k=0}^{N-1} f(k)g(m-k) \\
&= \sum_{k=0}^{N-1}\left[\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\mathcal{F}(n)e^{2\pi ikn/N}\right]\left[\frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\mathcal{G}(l)e^{2\pi i(m-k)l/N}\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\mathcal{F}(n)\sum_{l=0}^{N-1}\mathcal{G}(l)\sum_{k=0}^{N-1}e^{2\pi ikn/N}e^{2\pi i(m-k)l/N} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\mathcal{F}(n)\sum_{l=0}^{N-1}\mathcal{G}(l)e^{2\pi iml/N}\sum_{k=0}^{N-1}e^{2\pi ik(n-l)/N} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\mathcal{F}(n)\sum_{l=0}^{N-1}\mathcal{G}(l)e^{2\pi iml/N}\delta(n-l)N = \sum_{n=0}^{N-1}\left[\mathcal{F}(n)\cdot\mathcal{G}(n)\right]e^{2\pi imn/N}
\end{aligned}$$

□

DFT theorems

Stretch

If $f(m)$ is a 1D function of length N, $p \in \mathbb{N}$, and $stretch_p\{f\} = \{g\}$, where

$$g(n) = \begin{cases} f(n/p) & n = 0, p, 2p, \dots, (N-1)p \\ 0 & \text{otherwise} \end{cases}$$

then

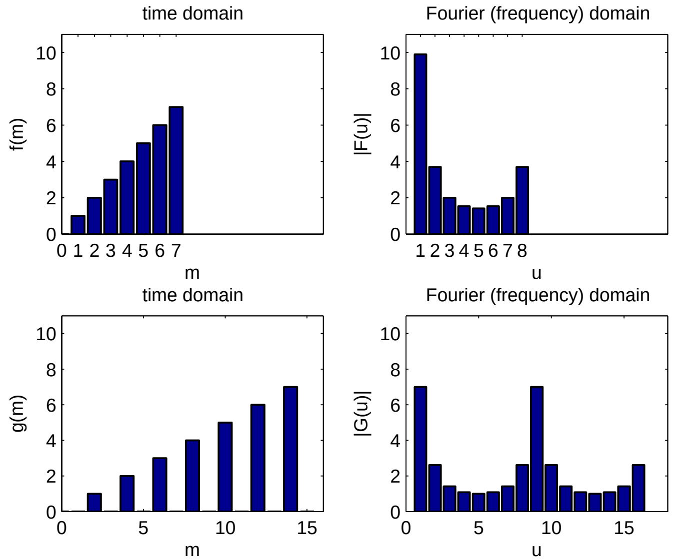
$$\mathcal{G}(k) = \frac{1}{\sqrt{p}} \begin{cases} \mathcal{F}(k) & k = 0, \dots, N-1 \\ \mathcal{F}(k-N) & k = N, \dots, 2N-1 \\ \vdots & \vdots \\ \mathcal{F}(k-(p-1)N) & k = (p-1)N, \dots, pN-1 \end{cases}$$

Notice: Stretch by a factor p in the time domain results in p-fold repetition of $\mathcal{F}(k)$ in the frequency domain.

DFT theorems

Stretch

An example of stretch for $p = 2$



DFT theorems

Periodicity of DFT

Statement:

$$\mathcal{F}(k+N) = \mathcal{F}(k)$$

Proof:

$$\begin{aligned} \mathcal{F}(k+N) &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m(k+N)}{N}} \\ &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}} e^{-\frac{2\pi i m N}{N}} \\ &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}} \\ &= \mathcal{F}(k) \end{aligned}$$

□

DFT theorems

Symmetry of real DFT

Statement:

$$\mathcal{F}(-k) = \overline{\mathcal{F}(k)}$$

Proof:

$$\begin{aligned} \mathcal{F}(-k) &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m(-k)}{N}} \\ &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{\frac{2\pi i m k}{N}} \\ &= \overline{\mathcal{F}(k)} \quad \text{iff} \quad f \in \mathbb{R}^N \end{aligned}$$

□

Fast Fourier Transform

Idea: N-point signal ($N = 2^m, m \in \mathbb{N}$) is decomposed into two $N/2$-point signals:

- one with all odd samples
- one with all even samples

Example:

1. input signal: $\{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ \} \supset ?$
2. $2\times$ simpler DFT: $\{0\ 2\ 4\ 6\} \supset \{A\ B\ C\ D\}$
   $2\times$ simpler DFT: $\{1\ 3\ 5\ 7\} \supset \{P\ Q\ R\ S\}$
3. stretch: $\{0\ 0\ 2\ 0\ 4\ 0\ 6\ 0\} \supset \frac{1}{\sqrt{2}}\{A\ B\ C\ D\ A\ B\ C\ D\}$
4. stretch: $\{1\ 0\ 3\ 0\ 5\ 0\ 7\ 0\ \} \supset \frac{1}{\sqrt{2}}\{P\ Q\ R\ S\ P\ Q\ R\ S\}$
   shift: $\{0\ 1\ 0\ 3\ 0\ 5\ 0\ 7\} \supset \frac{1}{\sqrt{2}}\{P\ \psi Q\ \psi^2 R\ \psi^3 S\ \psi^4 P\ \psi^5 Q\ \psi^6 R\ \psi^7 S\}$
5. linearity: $\{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\} = \{0\ 0\ 2\ 0\ 4\ 0\ 6\ 0\} + \{0\ 1\ 0\ 3\ 0\ 5\ 0\ 7\}$

$$\Downarrow$$

$$\{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\} \supset \frac{1}{\sqrt{2}}\{(A+P)\ \ (B+\psi Q)\ \ (C+\psi^2 R)\ \dots\}$$

Fast Fourier Transform

Derivation:

$$\begin{aligned}
\mathcal{F}(k) &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m) e^{-\frac{2\pi i (2m) k}{N}} + \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m+1) e^{-\frac{2\pi i (2m+1) k}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m) e^{-\frac{2\pi i m k}{N/2}} + e^{-\frac{2\pi i k}{N}} \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m+1) e^{-\frac{2\pi i m k}{N/2}} \\
&= \mathcal{F}^e(k) + \psi^k \mathcal{F}^o(k)
\end{aligned}$$

Notice: $\psi = e^{-\frac{2\pi i}{N}}$

Fast Fourier Transform

Idea: While it is possible repeat the division.

$$\begin{aligned}
\mathcal{F}(k) &\rightarrow \mathcal{F}^e(k), \mathcal{F}^o(k) \\
&\rightarrow \mathcal{F}^{ee}(k), \mathcal{F}^{eo}(k), \mathcal{F}^{oe}(k), \mathcal{F}^{oo}(k) \\
&\rightarrow \mathcal{F}^{eee}(k), \mathcal{F}^{eeo}(k), \mathcal{F}^{eoe}(k), \mathcal{F}^{eoo}(k), \mathcal{F}^{oee}(k), \mathcal{F}^{oeo}(k), \\
&\qquad \mathcal{F}^{ooe}(k), \mathcal{F}^{ooo}(k) \\
&\rightarrow \dots
\end{aligned}$$

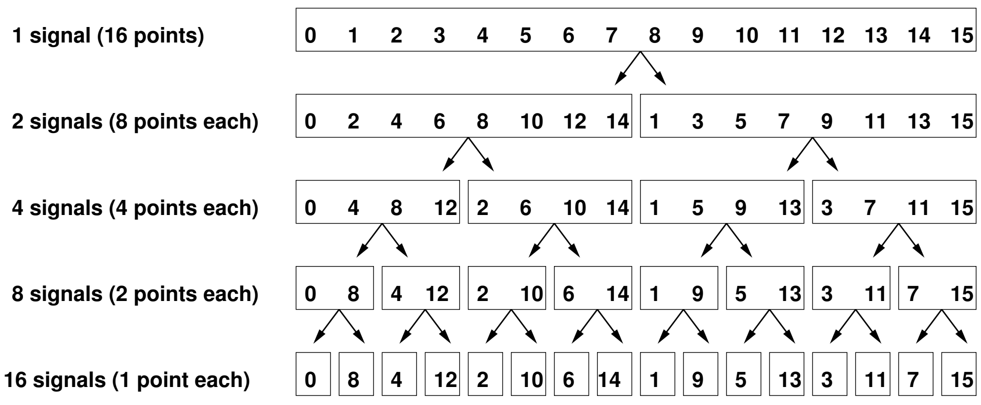
After $\log_2(n)$ divisions we have $\mathcal{F}^{eeeoe\dots eoeooee}(k)$ which is just one point long signal in Fourier domain.

You should know that: $\{X\} \supset \{X\}$

Exercise: What is the complexity of FFT?

Fast Fourier Transform

One more illustration



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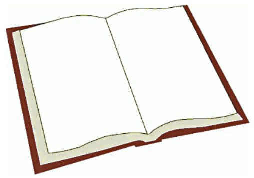

You should know the answers ...

- Express the discrete Fourier transform as a matrix multiplication. Derive this matrix.
- How many Fourier basis functions do we need if we transform the signal of length N into the frequency domain?
- Formulate the forward discrete Fourier transform. Explain all the variables and constants.
- What does DC and AC terms mean?
- Explain the meaning of one particular Fourier coefficient in inverse Fourier transform.
- What is the product of the projection of an even function into a sin wave?
- Why are the wide functions in time domain transformed into their narrow counterparts in frequency domain and vice versa?
- Derive FFT for a signal of length $3^m, m \in \mathbb{N}$.