Filters in Image Processing

Image Restoration

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Outline

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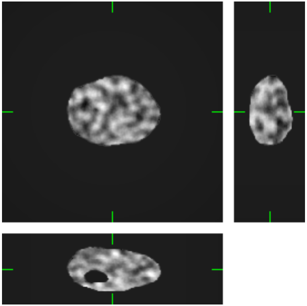
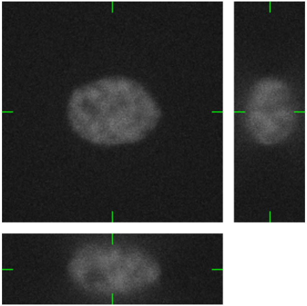
   - Blur
   - Noise
2. Restoration
   - Fundamentals
   - Unconstrained
   - Constrained
   - Iterative

Motivation

Images in optical microscopy are affected by blur and by noise. This blur is almost visible in z-axis. Images also suffer from noise due to low light intensities in confocal imaging.



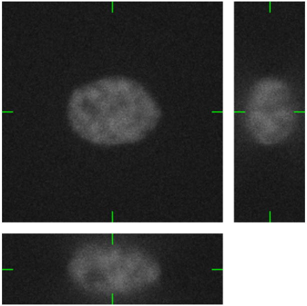
Cell at the start of optical path



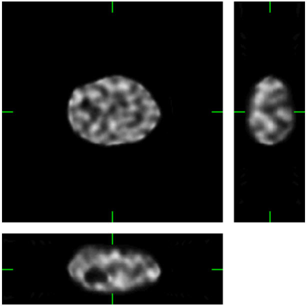
Cell after acquisition

Motivation

The task of image restoration $\equiv$ transforming the acquired image to its original form.



Acquired cell



Restored cell image

The blur can be described by Point Spread Function (PSF). PSF is response of optical setup to an infinitely small point source of light placed to the input. All the points are influenced by this function.

Blur can be caused by different sources:

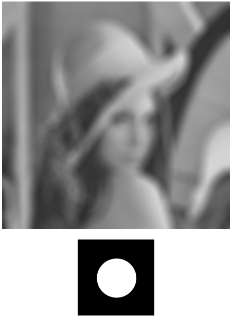
1. Move of the camera during acquisition
2. Defocus
3. Physical limits

Motivation

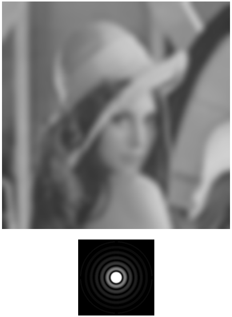
Image Blur

Examples

line - move



disc - defocused camera



Airy disc - physical limit

Noise is present in almost every real image. It can be caused by, for example:

- Environmental conditions during image acquisition (temperature)
- Quality of the sensing elements (hot pixels)
- Interference during image/data transmission

Noise is “a random change” of pixel values. Our interest is usually focused on the three basic types of noise:

- Additive noise (amplifiers)
- Impulse noise (hot/cold pixels in CCD)
- Poisson noise (photon-shot noise)

Examples

Additive noise

Impulse noise

Additive noise

The most common type of noise. Gray values and noise are independent:

$$g = f + n$$

where f is the original image, n is the noise, and g is the noisy image

Noise n may have different distributions:

- Gaussian distribution (amplifiers)
- Rayleigh distribution (radar)
- Exponential distribution (laser imaging)
- Gamma distribution (laser imaging)

Poisson Noise

- Important type of noise in CCD imaging (photon-shot noise, thermal noise)
- Poisson noise is not additive and depends on the signal.
- The noisy image f of the original one g is given by random Poisson process, which describes photon collection for each pixel position (i,j).
- Let photons occur at CCD pixel (i,j) with the average rate g_{ij} (photons/s) and let the original image is observed for t seconds. Then the probability that exactly X photons have occurred at pixel (i,j) is given by

$$p(X) = \frac{\lambda^X e^{-\lambda}}{X!}$$

where $\lambda = g_{ij} \cdot t$ and $X = 0, 1, 2, \ldots$

Restoration

Fundamentals

Digital image restoration tries to restore original image from acquired image WITH the knowledge of characteristics of degradations.

Examples of restorations:

- intensity correction
- chromatic aberration correction
- deconvolution

Deconvolution

Deconvolution is an inverse process to convolution. It tries to remove blur from image. This inverse process has to deal with noise as well.

Classification of deconvolution methods:

⇒ According to linearity of image processing:

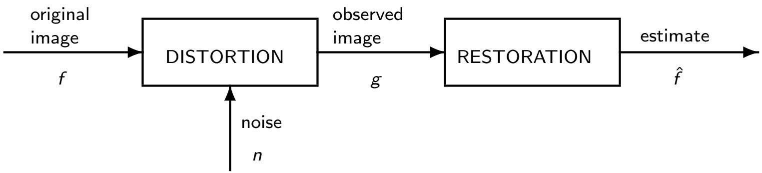
- Linear – linear filtering is performed
- Non-linear – non-linear filtering is performed

⇒ According to knowledge of PSF:

- Blind Deconvolution – PSF is unknown
- Non-blind Deconvolution – PSF is known

Deconvolution

Image degradation:

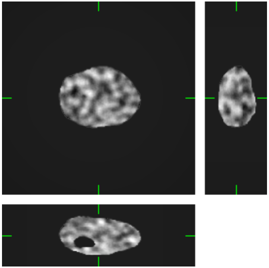


Deconvolution tries to invert degradation of an image. Such process is possible only in some cases. Sometimes, the image is irrecoverably damaged and we cannot restore most of details.

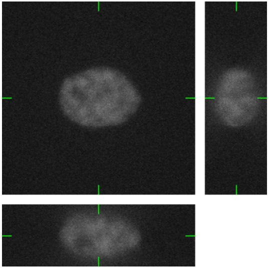
Notice: Blur removes some frequencies from the image. In this case, we cannot restore these frequencies.

Deconvolution
Example

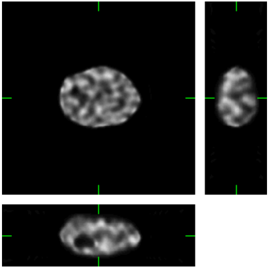
An example of blurred image may be an image acquired in confocal microscopy. Picture is blurred with the point-spread function (PSF) of microscope.



Cell at the start of optical path

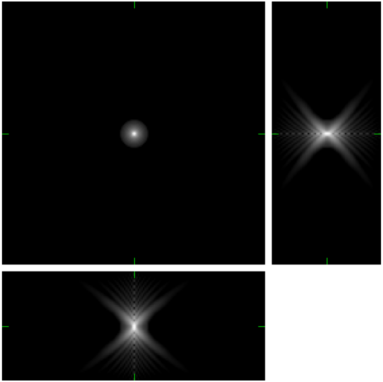


Cell after acquisition (blur + noise)



Acquired image after deconvolution

Point-spread function used in the previous slide:



Point-spread function of confocal microscope

Convolution × Matrix multiplication

Block circulant matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \quad C = A * B = \begin{bmatrix} -1 & -1 & 2 \\ -5 & -3 & 8 \\ -6 & -2 & 8 \end{bmatrix}$$

$$A_{ext} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_{ext} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_p = B_b \cdot A_p = \left[\begin{array}{ccc|ccc|ccc} -1 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 \\ \hline -2 & 0 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 1 & -1 \end{array}\right] \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -5 \\ -3 \\ 8 \\ -6 \\ -2 \\ 8 \end{bmatrix}$$

- C_p, A_p . . . linearized versions of C_{ext} and A_{ext}
- Matrix B_b . . . block circulant version of B_{ext}

Convolution × Matrix multiplication

Block circulant matrix

Any convolution

$$g = h * f$$

can be written as a matrix multiplication.
The above equation can be written as

$$g = Hf$$

where

- g and f are understood as linearized matrices, i.e. vectors
- H is block circulant version of matrix h

Notice: The math background and complexity is the same as in the case of the convolution. It is only a notation.

Unconstrained Restoration

Unconstrained restoration is a base method for image deconvolution. It is very fast and applicable to data without noise. This method assumes following image formation process:

$$g = Hf$$

g ... acquired image (including degradation)
H ... point spread function
f ... ideal image we are searching for
$\hat{f}$ ... our estimate of f

We seek to minimize the function:

$$W(\hat{f}) = \|e(\hat{f})\|^2 = \|g - H\hat{f}\|^2 = (g - H\hat{f})^T (g - H\hat{f})$$

where $\|a\| = \sqrt{a^T a}$ is the Euclidean norm of vector and $e(\hat{f}) = g - H\hat{f}$ is a vector of residuals.

Unconstrained Restoration

Setting the derivative of $W(\hat{f})$ to zero with respect to $\hat{f}$ produces:

$$\frac{\partial W(\hat{f})}{\partial \hat{f}} = -2H^T(g - H\hat{f}) = 0$$

and solving for $\hat{f}$ yields:

$$\hat{f} = H^{-1}g$$

which can be written (in the case of space-invariant PSF by using of convolution theorem):

$$\hat{f} = FT^{-1}\left\{\frac{FT(g)}{FT(h)}\right\}$$

Disadvantages:

- Problems with zero-amplitude frequencies.
- The approach doesn't deal with noise.
- Inverse matrix may not exists.

Cross-correlation

Cross-correlation function R_{fg} indicates the relative degree to which two functions f and g agree for various amounts of misalignment. It is given by

$$R_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)g(t+\tau)dt$$

Auto-correlation

Auto-correlation is special case of cross-correlation:

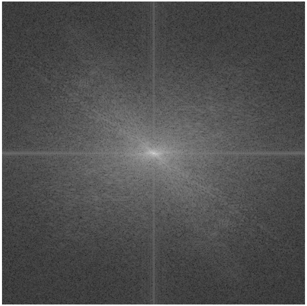
$$R_{f}(\tau) = \int_{-\infty}^{\infty} f(t)f(t+\tau)dt$$

Power spectrum

Power spectrum $\mathcal{P}$ of signal f is the FT of autocorrelation of f.

$$FT[R_f(\tau)] = FT[f(t) * f(-t)] = \mathcal{F}(k)\mathcal{F}(-k) = \mathcal{F}(k)\mathcal{F}^*(k) = |\mathcal{F}(k)|^2 = \mathcal{P}_f(k)$$

Lena



Power spectra of Lena

We are searching for filter W that suppresses noise and keeps the signal quality:

$$\hat{f} = W * (f + n) = W * s$$

- f is noise-free input
- W is unknown filter
- n is additive noise
- $\hat{f}$ is signal $s = f + n$ filtered with W

We try to minimize:

$$MSE(W) = \int_{-\infty}^{\infty} e^2(t)dt = \int_{-\infty}^{\infty} \left(f(t) - \hat{f}(t)\right)^2 dt = \ldots$$

$$MSE(W) = \int_{-\infty}^{\infty} \left(f(t) - \hat{f}(t)\right)^2 dt$$

$$= \int_{-\infty}^{\infty} \left[f^2(t) - 2f(t)\hat{f}(t) + \hat{f}^2(t)\right] dt = T_1 + T_2 + T_3$$

where

$$T1 = \int_{-\infty}^{\infty} f(t)^2 dt = R_f(0)$$

$$T2 = -2 \int_{-\infty}^{\infty} f(t)\hat{f}(t)dt = -2 \int_{-\infty}^{\infty} f(t)[(W * s)(t)]dt = -2 \int_{-\infty}^{\infty} W(t)R_{fs}(t)dt$$

$$T3 = \int_{-\infty}^{\infty} \hat{f}^2(t)dt = \int_{-\infty}^{\infty} (W * s)(t)(W * s)(t)dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(t)W(\tau)R_s(\tau - t)d\tau dt$$

Wiener Filtering

Derivation

$$MSE(W) = R_f(0) - 2\int_{-\infty}^{\infty} W(t)R_{fs}(t)dt + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t)W(\tau)R_s(\tau - t)d\tau dt$$

We try to minimize MSE(W), i.e.

$$\frac{\partial MSE(W)}{\partial W} \to 0$$

Solution

$$R_{fs} = W * R_s \quad \to \mathrm{FT} \to \quad \mathcal{P}_{fs} = FT(W) \cdot \mathcal{P}_s \quad \to \quad FT(W) = \frac{\mathcal{P}_{fs}}{\mathcal{P}_s}$$

Wiener Filtering

Provided noise and signal are NOT correlated the transfer function of filter 'W' minimizing MSE is defined:

$$FT(W) = \frac{\mathcal{P}_{fs}}{\mathcal{P}_s} = \frac{\mathcal{P}_f}{\mathcal{P}_f + \mathcal{P}_n} = \frac{|FT(f)|^2}{|FT(f)|^2 + |FT(n)|^2}$$

where:

- P_f is power spectrum of noise-free signal
- P_n is power spectrum of noise

Notice: 'W' is called Wiener filter.

Wiener Deconvolution

Unconstrained restoration followed by Wiener filtering we get optimal deconvolution respecting the MSE condition:

$$\hat{f} = FT^{-1}\left(\frac{FT(g)}{FT(h)}\frac{\mathcal{P}_f}{\mathcal{P}_f + \mathcal{P}_k}\right)$$

where

$$\mathcal{P}_k = \left|\frac{FT(n)}{FT(h)}\right|^2$$

is a cross power spectrum of noise and PSF.

There exists more practical version of Wiener deconvolution:

$$\hat{f} = FT^{-1}\left(\frac{FT(g)}{FT(h)}\frac{\mathcal{P}_g - \mathcal{P}_n}{\mathcal{P}_g}\right)$$

Wiener Deconvolution

An example

Lena

Deconvolved Lena

Constrained Least Square Restoration

Idea

To suppress noise in the result of deconvolution we should impose some constraints to solution. Such constraint can be the convolution of result with some convolution kernel and the minimization of power of this image. The task is to minimize:

$$W(\hat{f}) = \|Q\hat{f}\|^2 + \lambda(\|g - H\hat{f}\|^2 - \|n\|^2)$$

$$g - Hf = n$$

where

- Q ... convolution kernel, which imposes some constraint on resulting image $\hat{f}$
- λ ... Lagrange multiplier (magic factor)

Example: Concerning Q, we can use Laplace filter which boosts high frequencies (noise).

Constrained Restoration

Now we need to solve equation:

$$\frac{\partial W(\hat{f})}{\partial \hat{f}} = 2Q^T Q\hat{f} - 2\lambda H^T(g - H\hat{f}) = 0$$

Solving for $\hat{f}$ then yields:

$$\hat{f} = (\lambda H^T H + Q^T Q)^{-1} \lambda H^T g$$

Constrained Least Square Deconvolution

An example

Lena

Deconvolved Lena

Iterative Methods

In the most cases it is not possible to find solution of problem of restoration by simple linear filtering. Hence there exist different methods for restoration, which are iterative. This methods iteratively improve initial estimate until it is acceptable.

Most common iterative methods:

- EM-MLE (Expectation Maximization – Maximum Likelihood Estimation)
- ICTM (Iterative Constrained Tikhonov-Miller algorithm)

Used criteria:

- number of iterations
- relative change between two iterations
- difference between blurred estimate and input

EM-MLE

EM-MLE is one of the best methods for restoration of images degraded by noise with Poisson statistics. Its principle consists in maximizing certain functional. Such maximization is performed by Expectation-Maximization method, which is iterative numerical method. The functional which is maximized is likelihood functional and is expressed by:

$$L(\hat{f}) = \prod_j p(x) = \prod_j \frac{[\mu(u_j)]^{N_j} e^{-\mu(u_j)}}{N_j!},$$

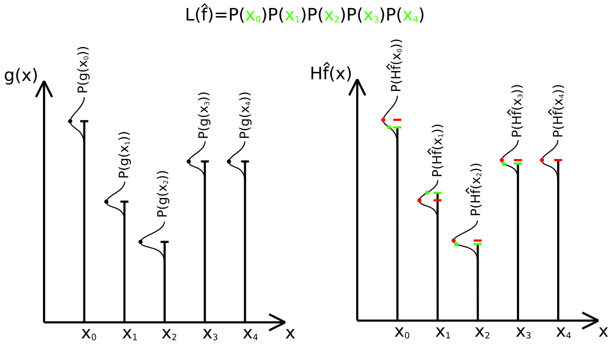
where

- $p(x)$ is a probability density function following Poisson distribution
- N_j is j-th point value of acquired image
- $\mu(u_j)$ is blurred version of our estimate $(H\hat{f})$.

Notice: EM-MLE method is also known as Richardson-Lucy method.

EM-MLE

Illustration



On the left is acquired image with Poisson distributions on values of $g(x)$. On the right, you can see the same distributions, but used probabilities values (green spots) are from positions of $(H\hat{f})(x)$ (green lines). Product of this probabilities is likelihood value.

EM-MLE

After derivation of this method we get simple expression for one iteration:

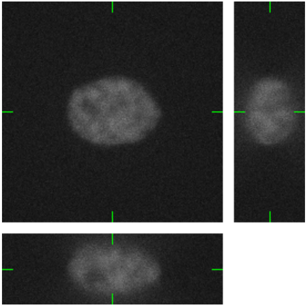
$$f^{k+1} = f^k H^T \left(\frac{g}{Hf^k + b} \right),$$

where

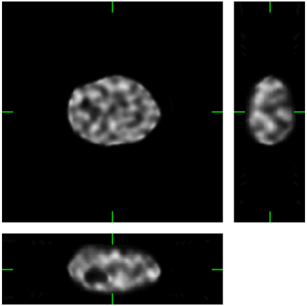
- b is the known background
- Hf^k and $H^T()$ are matrix-vector multiplications
- all other operations are point-wise

EM-MLE

Example



Acquired cell



Restored cell image

ICTM

ICTM method considers additive noise with Gaussian distribution. Due to iterative form, we can impose non-linear constraints to result of deconvolution. One of the base constraints is non-negativity of resulting intensities. This cannot be achieved by non-iterative method.
In this method we seek to minimize functional of the form:

$$W(\hat{f}) = \frac{1}{2}(\|H\hat{f} - g\|^2 + \gamma\|Q\hat{f}\|^2).$$

Notice: The functional of this form is solvable by algorithm of conjugate gradients (see numerical methods – FI:PV027 Optimization).

Computation consists in the following steps:

$$f^{k+1} = P(f^k + \alpha^k d^k),$$

where d^k is direction of $k-th$ step, α^k is size of the $k-th$ step, f^k is result of the last iteration and $P()$ is projection operator, which clips values to non-negative values. Direction d^{k+1} is computed in this way:

$$d^{k+1} = \frac{\|r^k\|^2}{\|r^{k-1}\|^2} d^k - r^k,$$

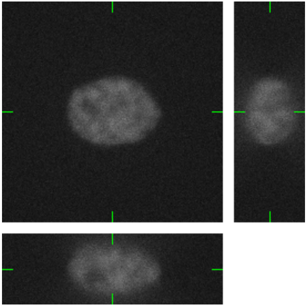
where $r^k = Af_k - b$, $A = H^T H + \gamma Q^T Q$ and $b = H^T g$.
Termination of algorithm:

$$threshold > \left| \frac{W^{i+1} - W^i}{W^{i+1}} \right|,$$

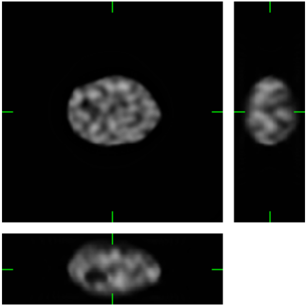
where W^i is value of $W()$ at iteration i.

ICTM

An example



Acquired cell



Restored cell image

Bibliography

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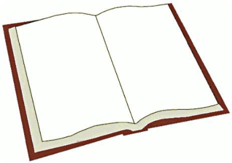
- Castleman, R. C., Digital Image Processing, Prentice Hall, 1996
- Verveer, P. J., Computational and Optical Methods for Improving Resolution and Signal Quality in Fluorescence Microscopy, PhD thesis, Technical University Delft, 1998



You should know the answers . . .

- Explain the difference between a *constrained* and *unconstrained* image restoration.
- Explain the equation $g = H * f + n$.
- Explain why we cannot invert the convolution theorem to eliminate the consequences of convolution with known PSF.
- Explain the meaning of the multiplication ($\prod$) in EM-MLE algorithm.
- What is a difference between Wiener filtering and Wiener deconvolution?
- How do we get $\mathcal{P}_n$ in order we could perform the Wiener deconvolution?
- Are you able to implement *Constrained least square restoration* in your favourite programming language?
- How do we stop iterative restoration methods?