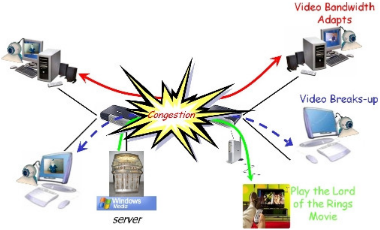
Selfish Routing
Congestion Games

Selfish Routing – Motivation

Many agents want to use shared resources

Each of them is selfish and rational
(i.e. maximizes his profit)



Examples: Users of a computer network, drivers on roads

How they are going to behave?

How much is lost by letting agents behave selfishly on their own?

Example: Routing in Computer Networks

Imagine a computer network, i.e., computers connected by links.

There are several users, each user wants to route packets from a *source* computer z_i to a *target* computer t_i. For this, each user i needs to choose a path in the network from z_i to t_i.

We assume that the more agents try to route their messages through the same link, the more the link gets congested and the more costly the transmission is.

Now assume that the users are selfish and try to minimize their cost (typically transmission time). How would they behave?

Atomic Routing Games

The network routing can be formalized using an atomic routing game that consists of

- a directed multi-graph $G = (V, E, \delta)$,

  Here V is a set of vertices, E is a set of edges, $\delta : E \to V \times V$ so that if $\delta(e) = (u, v)$ then e leads from u to v. The multigraph G models the network.

- n pairs of source-target vertices $(z_1, t_1), \dots, (z_n, t_n)$ where $z_1, \dots, z_n, t_1, \dots, t_n \in V$,

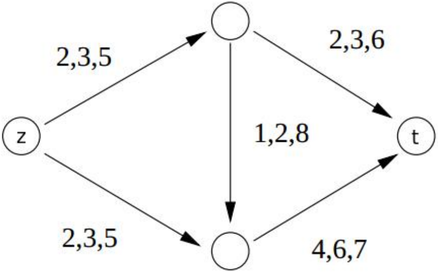
  (Each pair (z_i, t_i) corresponds to a user who wants to route from z_i to t_i)

- for each $e \in E$ a cost function $c_e : \mathbb{N} \to \mathbb{R}$ such that $c_e(m)$ is the cost of routing through the link e if the amount of traffic through e is m.

A pure strategy s_i of player i is a simple path from z_i to t_i, the payoff is *minus* the sum of the costs of the links on the path.

Note that each routing game can be seen as a strategic-form game.

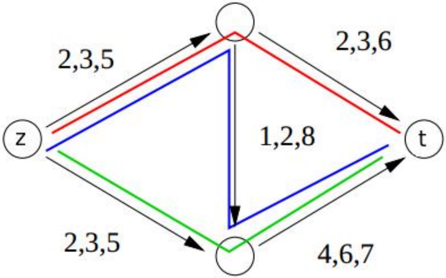
Atomic Routing Games



Here we assume at most three users. Each edge is labeled by the cost if one, two, or all three users route through the edge, respectively.

Here we consider a symmetric case with three users, each has the source z and target t.

Atomic Routing Games



Here, e.g., the red user pays $3 + 2 = 5$:

- 3 for the first step from z (he shares the edge with the blue one)
- 2 for the second step to t (he is the only user of the edge)

Solving Congestion Games

We consider the following questions:

- Are there pure strategy Nash equilibria?
- Can the agents "learn" to use the network?

$$\begin{pmatrix} -1,1 & 1,-1 & -2,-2 \\ 1,-1 & -1,1 & -2,-2 \\ -2,-2 & -2,-2 & 2,2 \end{pmatrix}$$

Learning: Myopic Best-Response

Given a pure strategy profile $s = (s_1, \dots, s_n)$, suppose that some player i has an alternative strategy s_i' such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$. Player i can switch (unilaterally) from s_i to s_i' improving thus his payoff. Iterating such *improvement steps*, we obtain the following:

Myopic best response procedure:

- Start with an arbitrary pure strategy profile $s = (s_1, \dots, s_n)$.
- While there exists a player i for whom s_i is *not a best response* to s_{-i} do
  - $s_i' :=$ a best-response by player i to s_{-i}
  - $s := (s_i', s_{-i})$
- return s

By definition, if the myopic best response terminates, the resulting strategy profile s is a Nash equilibrium.

There is a strategic-form game where it does not terminate.

Theorem 1

For every routing game, the myopic best response terminates in a Nash equilibrium for an arbitrary starting pure strategy profile.

Non-Atomic Selfish Routing

- So far we have considered situations where each player (user, driver) has enough "weight" to explicitly influence payoffs of others (so a deviation of one player causes changes in payoffs of other players).
- In many applications, especially in the case of highway traffic problems, individual drivers have negligible influence on each other. What matters is a "distribution" of drivers on highways.
- To model such situations we use *non-atomic routing games* that can be seen as a limiting case of atomic selfish routing with the number of players going to ∞.

Non-Atomic Routing Games

A *Non-Atomic Routing Game* consists of

- a directed multigraph $G = (V, E, \delta)$,
- n source-target pairs $(z_1, t_1), \ldots, (z_n, t_n)$,
- for each $i = 1, \ldots, n$, the *amount of traffic* $\mu_i \in \mathbb{R}_{\geq 0}$ from z_i to t_i,
- for each $e \in E$ a cost function $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $c_e(x)$ is the cost of routing through the link e if the amount of traffic on e is $x \in \mathbb{R}_{\geq 0}$.

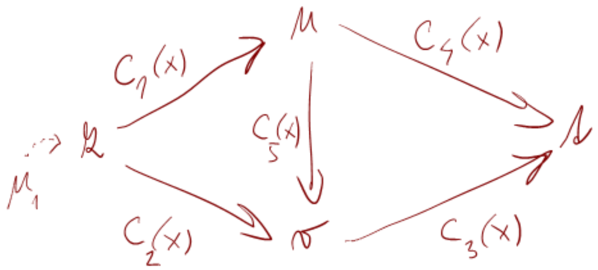
For $i = 1, \ldots, n$, let $\mathcal{P}_i$ be the set of all simple paths from z_i to t_i.

Intuitively, there are uncountably many players, represented by $[0, \mu_i]$, going from z_i to t_i, each player chooses his path so that his total cost is minimized.

Assume that $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for $i \neq j$.

(This also implies that for all $i \neq j$ we have that either $z_i \neq z_j$, or $t_i \neq t_j$.)
Denote by $\mathcal{P}$ the set of all "relevant" simple paths $\bigcup_{i=1}^{n} \mathcal{P}_i$.

Question: What is a "stable" distribution of the traffic among paths of $\mathcal{P}$?



$d(r \to u \to v \to s) = \frac{\mu_1}{6}$

$d(r \to u \to s) = \frac{2}{3}\mu_1$

$d(r \to v \to s) = \frac{\mu_1}{6}$

$g(d, v-s) = \frac{\mu_1}{3}$

$u(d, r \to u \to v \to s) = -\left(c_1\left(\frac{5}{6}\mu_1\right) + c_5\left(\frac{\mu_1}{6}\right) + c_3\left(\frac{\mu_1}{3}\right)\right)$

Non-Atomic Routing Games

A *traffic distribution* d is a function $d : \mathcal{P} \to \mathbb{R}_{\geq 0}$ such that $\sum_{p \in \mathcal{P}_i} d(p) = \mu_i$. Denote by D the set of all traffic distributions.

Let us fix a traffic distribution $d \in D$.

Given an edge $e \in E$, we denote by $g(d, e)$ the *amount of congestion on the edge e* :

$$g(d, e) = \sum_{p \in \mathcal{P} \,:\, e \in p} d(p)$$

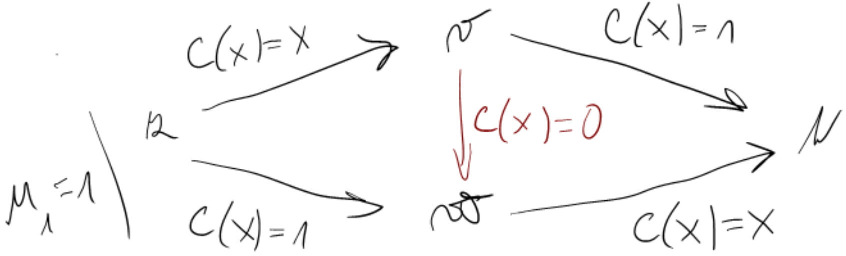
Given $p \in \mathcal{P}$, the *payoff for players routing through $p \in \mathcal{P}$* is defined by

$$u(d, p) = -\sum_{e \in p} c_e(g(d, e))$$

Definition 2

A traffic distribution $d \in D$ is a Nash equilibrium if for every $i = 1, \ldots, n$ and every path $p \in \mathcal{P}_i$ such that $d(p) > 0$ the following holds:

$$u(d, p) \geq u(d, p') \text{ for all } p' \in \mathcal{P}_i$$



$d(s \to v \to t) = \frac{1}{2}$ $\quad d(s \to w \to t) = \frac{1}{2}$ is a NE, $C(d) = \frac{3}{2}$

$d(s \to v \to w \to t) = 1$

$C(d) = 2$

Price of Anarchy

Theorem 3

Every non-atomic routing game has a Nash equilibrium.

We define a *social cost* of a traffic distribution d by

$$C(d) = \sum_{p \in \mathcal{P}} d(p) \cdot (-u(d,p)) = \sum_{p \in \mathcal{P}} d(p) \cdot \sum_{e \in p} c_e(g(d,e))$$

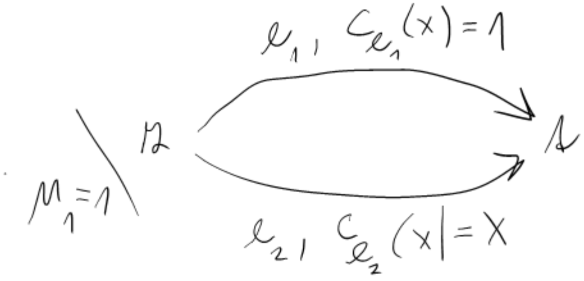
Theorem 4

All Nash equilibria in non-atomic routing games have the same social cost.

A *price of anarchy* is defined by

$$PoA = \frac{C(d^*)}{\min_d C(d)} \quad \text{where } d^* \text{ is a (arbitrary) Nash equilibrium}$$

Intuitively, PoA is the proportion of additional social cost that is incurred because of agents' self-interested behavior.



$d(e_1) = 0$
$d(e_2) = 1$

$d(e_1) = \frac{1}{2}$
$d(e_2) = \frac{1}{2}$

$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

$PoA = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

Price of Anarchy

Theorem 5 (Roughgarden-Tardos'2000)

For all non-atomic routing games with linear cost functions holds

$$PoA \leq \frac{4}{3}$$

and this bound is tight (e.g. the Pigou's example).

The price of anarchy can be defined also for atomic routing games:

$$PoA_{atom} := \frac{\max_{s^* \text{ is } NE} \sum_{i=1}^{n}(-u_i(s^*))}{\min_{s \in S} \sum_{i=1}^{n}(-u_i(s))}$$

(Intuitively, $\sum_{i=1}^{n}(-u_i(s))$ is the total amount paid by all players playing the strategy profile s.)

Theorem 6 (Christodoulou-Koutsoupias'2005)

For all atomic routing games with linear cost functions holds

$$PoA_{atom} \leq \frac{5}{2}$$

(which is again tight, just like $\frac{4}{3}$ for non-atomic routing.)

Braess's Paradox

For an example see the green board.

Real-world occurences (Wikipedia):

- In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.
- In Stuttgart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly built road was closed for traffic again.
- In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.
- In 2012, scientists at the Max Planck Institute for Dynamics and Self-Organization demonstrated through computational modeling the potential for this phenomenon to occur in power transmission networks where power generation is decentralized.
- In 2012, a team of researchers published in Physical Review Letters a paper showing that Braess paradox may occur in mesoscopic electron systems. They showed that adding a path for electrons in a nanoscopic network paradoxically reduced its conductance.