Tutorial V

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Assymetric Cryptography

- Basics of number theory
- RSA encryption
- Diffie Hellman key exchange
- Knapsack Cryptosystem

Basic Number theory

$\mathbb{Z}_n \rightsquigarrow$ set of all remainders after division by n

$+ \bmod n$, $(\mathbb{Z}_n, +)$ is a group

$\mathbb{Z}_n^* \rightsquigarrow$ set of all non-zero remainders after division by n

$(\mathbb{Z}_n^*, \cdot \bmod n) \rightsquigarrow$ group for prime n (multiplicative inverses exist)

$\rightsquigarrow$ for general n monoid (not all multiplicative inverses exist)

$$\frac{a}{b} \bmod n = a \cdot b^{-1} \bmod n \quad \left(b^{-1} \text{ exists iff } \gcd(b, n) = 1 \right)$$

$$\cancel{\frac{5}{3} \bmod 7 \neq 1{,}666}$$ ← INCORRECT

$$5 \cdot 3^{-1} \bmod 7 \quad (3^{-1} = 5,\ 3 \cdot 5 = 15 \equiv 1 \bmod 7)$$

$$5 \cdot 5 \bmod 7$$

4 mod 7

How to calculate inverses mod n?

Euclid's algorithm → algorithm to calculate gcd(a,b) for any $(a,b) \in \mathbb{Z}$

Bézout's identity → for a,b: gcd(a,b) = 1

$\exists\, x,y$ s.t. $ax + by = 1$

Extended Euclid's algorithm → algorithm to calculate x,y from Bézouts identity

$$ax + by = 1$$

$$ax = 1 - by \quad | \bmod b$$

$$ax \equiv 1 \quad (\bmod b)$$

$$a^{-1} \equiv x \quad \bmod b$$

$$\Rightarrow ax \equiv 1 - 0$$

Find $\gcd(17, 3) = 1$ (a = 17, b = 3)

$17 : 3 = 5 \quad rm\ 2$ → $2 = 17 - 3 \cdot 5$

$3 : 2 = 1 \quad rm\ 1$ ← last non-zero remainder is gcd(a,b) (17, 3)

→ $1 = 3 - 2 \cdot 1$

$2 : 1 = 2 \quad rm\ 0$

$2 : 1 = 2 \quad \text{rm } 0$

$1 = 3 - \boxed{2} \cdot 1$

$1 = 3 - (17 - 3 \cdot 5) \cdot 1$

$1 = 3 - 17 + 3 \cdot 5$

$1 = 3 \cdot 6 - 17 \cdot 1$

$y = 6$

$x = -1$

$3^{-1} = 6 \mod 17 \qquad 3 \cdot 6 = 18 \equiv 1 \mod 17$

$17^{-1} = 2 \mod 3 \qquad 2 \cdot 17 = 34 \equiv 1 \mod 3$

Modular exponentiation

$a^b \mod n$

$2^{303} \mod 3$

$\cancel{2^{303 \bmod 3} = 2^0 = 1 \mod 3}$ ← INCORRECT

$2^{303} = 2^{303 \bmod 2} = 2^1 = 2 \mod 3$

For a, with $\gcd(a,n) = 1$

$a^b \mod n = a^{b \bmod \varphi(n)} \mod n$

Euler's Totient theorem

for a, n: $a < n$, $\gcd(a,n) = 1$

$a^{\varphi(n)} \equiv 1 \mod n$

$\varphi(n)$ – Euler's theta function

– Euler's totient function

= number of $a < n$

$\gcd(a,n) = 1$

$\varphi(p) = p - 1$ for prime p

$\varphi(m \cdot n) = \varphi(m) \cdot \varphi(n) \cdot \frac{d}{\varphi(d)}$

where $\gcd(m,n) = d$

p, q are prime

$\varphi(p \cdot q) = \varphi(p) \cdot \varphi(q) \cdot \frac{1}{\varphi(1)}$

$$a^b \mod n = a^{b \mod \varphi(n)} \mod n$$

$$\varphi(p\cdot q) = \varphi(p)\cdot\varphi(q)\cdot\frac{1}{\varphi(1)} = (p-1)(q-1)$$

$$\underbrace{\overbrace{a\cdot a\cdot a\cdot\ a\cdot}^{\varphi(n)=1}\ \overbrace{-\ \cdot\ \cdot\ \cdot}^{\varphi(n)=1}\ \cdots\ \overbrace{a}^{b \mod \varphi(n)}}_{b\text{- times}}$$

For prime n you recover Fermat's little theorem

$$\forall a < n$$

$$a^{n-1} \equiv 1 \mod n \qquad (n-1 = \varphi(n))$$

$$a^{b \mod n-1} \mod n$$

Important problems in assymetric cryptography

Factorization easy: Given a, b find c, s.t. $c = a\cdot b$

hard: Given c find a, b, s.t. $c = a\cdot b$

Essentially trying all divisors between 2 and $\sqrt{c}$ is the best algorithm we know.

$\boxed{2048\text{- bits}}$ $\quad c \mathrel{\hat{=}} 2^{2048}$ $\quad \boxed{\sqrt{c} = 2^{1024}}$

Number of protons in the Universe $\approx 2^{300}$

Discrete logarithm problem

easy: given a, b and n
calculate $a^b \bmod n$

hard: given c, a, n calculate b,
such that $c = a^b \bmod n$ to $\{1, \ldots, \varphi(n)\}$

$b = \log_a c \bmod n$ if $\gcd(a, n) = 1$

RSA encryption

Private: p, q → two large primes $n = p \cdot q$

$d = e^{-1} \bmod \bmod \underbrace{(p-1)(q-1)}_{\varphi(n)}$

Public: e, n

Encryption of message $w < n$ if w is larger, this needs to be done in blocks

$\boxed{172}\boxed{459}$ $1000 < n < 9999$

$$C = w^e \bmod n$$

Decryption of ciphertext c

$$w = c^d \bmod n$$
$$= (w^e)^d \bmod n$$
$$= w^{e \cdot d} \bmod n$$
$$= w^{e \cdot d \bmod \varphi(n)} \bmod n$$
$$= w^1 \bmod n$$

! $\gcd(w, n) = 1$

What can an adversary do if they do not know p, q, d ?

1.) Factorize n $\Rightarrow$ p and q $\Rightarrow$ $\varphi(n) = (p-1)(q-1)$ $\Rightarrow$ calculate $d = e^{-1} \bmod n$

Hard

2.) Can I find an efficient algorithm to calculate $\varphi(n)$ without factoring n?

Then we can factor n like this

$$\left.\begin{aligned} p \cdot q &= n \\ (p-1)\cdot(q-1) &= \varphi(n) \end{aligned}\right| \longrightarrow \text{easy to solve system of equations}$$

3.) $e, n \rightarrow d$ RSA problem

This is hard (we do not know an efficient algorithm)
but probably (we do not know an efficient reduction)
not as hard as factoring.

Other RSA weaknesses

For known (w, c) pairs ($w^e = c \bmod n$) you can find other pairs

(w^2, c^2) is also a valid pair

$$(w^2)^e \equiv w^{2e} = w^e \cdot w^e = c \cdot c = c^2 \bmod n$$

Whenever you see c^2 in a channel and you know (w, c) is a valid pair you also know that c^2 decrypts as w^2.

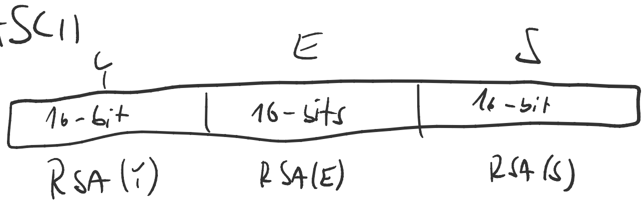
(w^n, c^n) is a valid pair for any c.

if (w_1, c_1) and (w_2, c_2) are two valid pairs then also

$(w_1 \cdot w_2, c_1 c_2)$ is a valid pair

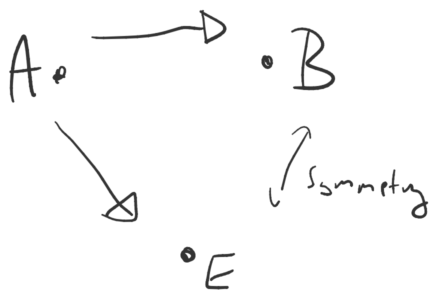
$$(w_1 \cdot w_2)^e = w_1^e \cdot w_2^e = c_1 \cdot c_2 \mod n$$

TEXT → ASCII



THIS IS JUST A MONOALPHABETIC SUBSTITUTION
(NOT SECURE)

DIFFIE-HELLMAN KEY DISTRIBUTION



Prerequisities

p - is a large prime

$\boxed{q \in \mathbb{Z}_p^* \text{ with a large order}}$

$a^{p-1} = 1 \mod p$

$1 \quad 1^2 \quad 1^3$
$= 1 \quad = 1 \quad = 1$ order $(1) = 1$

$(p-1)^1 \quad (p-1)^2 \quad (p-1)^3$
$= p-1 \quad = 1 \quad = p-1$ order $(p-1) = 2$

$A \to B$

$A = q^x \mod p$ these are chosen randomly

$B \to A$

$B = q^y \mod p$

A calculates

A calculates

$k = B^x \bmod p = g^{xy} \bmod p$

B calculates

$k = A^y \bmod p = g^{yx} \bmod p$

$p-1 \quad 1 \quad p-1$

$\log_{p-1} c \bmod n$

What can the adversary do?

1.) calculate $x = \log_g A \bmod p$
$y = \log_g B \bmod p$ $\to$ Discrete logarithm problem
HARD

$k = g^{xy} \bmod p$

2.) given g^x and g^y calculate $g^{xy} \bmod p$

$\boxed{\text{DH 1}}$ $\to$ believed to be hard

KNAPSACK CRYPTOSYSTEM

NP-complete problem (based on)

given $(x_1, \dots, x_n)$ $x_i \in \mathbb{Z}_m$ for large n

and a constant c

find $b \in \{0,1\}^n$, such that

$\vec{x} \cdot \vec{b} = c \bmod m$

$\vec{x} \cdot \vec{b} = c \mod m$

Easy instance - superincreasing vectors

X is superincreasing $\forall i \; x_i > \sum_{j<i} x_j$

$x_2 > x_1$
$x_3 > x_1 + x_2$
$x_4 > x_1 + x_2 + x_3$

for $c < 2x_n$

an instance with superincreasing X and c is easy.

Public information X, m $\qquad m > 2x_n > \sum_i x_i$

Private information u invertible mod m and $X' = u^{-1} \cdot X \mod m$ where X' is superincreasing

encryption

$w \in \{0,1\}^n$

$c = w \cdot X \mod m$

(c, X, m) - are a subset sum instance

decryption calculate $u^{-1} \cdot c = c' \mod p$

then solve subsetsum instance with (c', x')

$c = w \cdot X \qquad / u^{-1}$

$c u^{-1} = w \cdot \underbrace{u^{-1} \cdot X} \mod m$

$c' = w \cdot X'$

$X' = (1, 3, 7, 13, 29, 59, 127)$

$X = \underset{u}{\underset{\downarrow}{155}} \cdot X' \mod \underset{m}{\underset{\downarrow}{257}}$

$= (155, 208, 57, 216, 126, 150, 153)$

$w = (0111011)$

$C = w \cdot X = 208 + 57 + 216 + 150 + 153 = 784 \mod 257$

$= 13 \mod 257$

$13, X$

$13 \cdot u^{-1} = 13 \cdot 155^{-1} \mod 257$

$= 13 \cdot 194 \mod 257$

$= 1940 + 600 - 18 = 2522 \mod 257$

$= -48 \mod 257$

$= 209 \mod 257$

$(209, X')$

$(0\ 1\ 1\ 1\ 0\ 1\ 1) = w$

$X' = (1, 3, 7, 13, 29, 59, 127)$

$209 - 127 = 82$

$82 - 59 = 23$

$23 - 13 = 10$