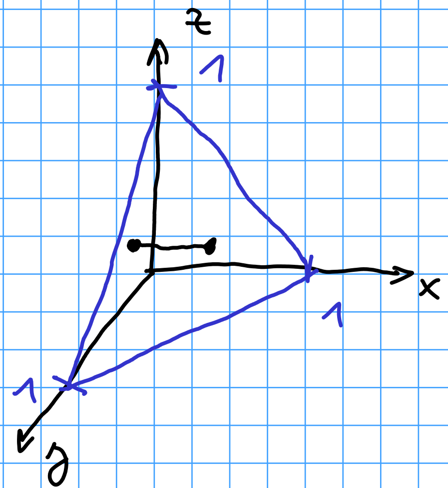
• Spočítejte integrál $\iiint_M f(x,y,z)\,dx\,dy\,dz$,

kde M je množina ohraničená rovinami $x=0, y=0, z=0$ (v 1. oktantu)

a rovinou $x+y+z=1$.

$x = 1-y-z$



$z \in [0, 1]$

$y \in [0, 1-z]$

$x \in [0, 1-y-z]$

$$\iiint_M f(x,y,z)\,dx\,dy\,dz =$$

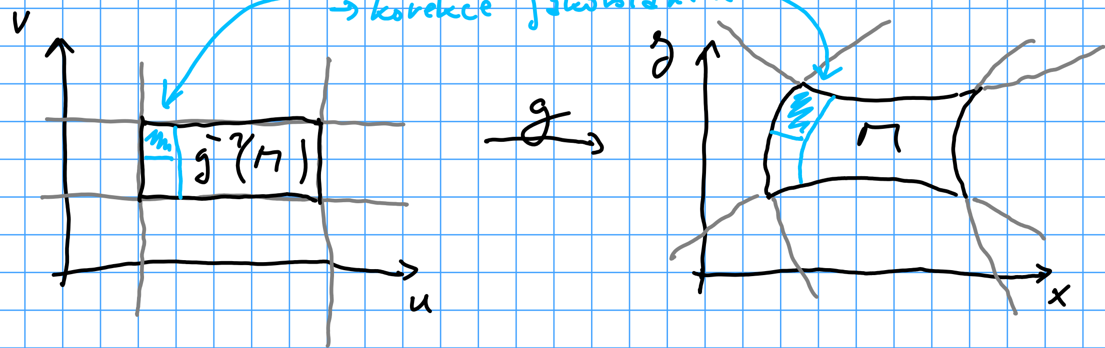
$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} f(x,y,z)\,dx\,dy\,dz$$

Transformace souřadnic

$$\iint_M f(x,y)\,dx\,dy = \iint_{g^{-1}(M)} f\big(x(u,v), y(u,v)\big)\cdot \big|J(u,v)\big|\,du\,dv$$

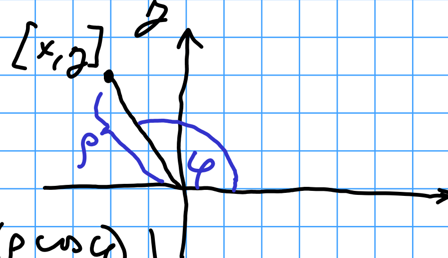
jakobián

různý obsah „čtverečku"
→ korekce jakobiánem



$$g: \begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array} \qquad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Transformace do pol. souřadnic

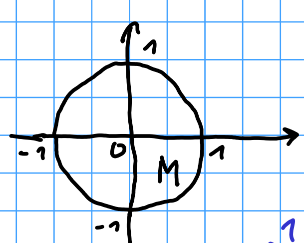


$$x = \rho\cdot\cos\varphi \qquad y = \rho\cdot\sin\varphi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial \rho}(\rho\cos\varphi) & \frac{\partial}{\partial \varphi}(\rho\cos\varphi) \\ \frac{\partial}{\partial \rho}(\rho\sin\varphi) & \frac{\partial}{\partial \varphi}(\rho\sin\varphi) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho$$

- Spočítejte integrál $I = \iint_M \sin\left(\sqrt{x^2+y^2}\right)dx\,dy$, kde $M: x^2+y^2 \le 1$



$\rho \in [0,1]$

$\varphi \in [0,2\pi]$

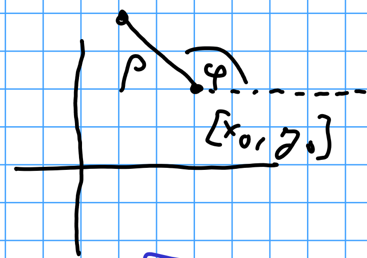
$$= \int_0^1\int_0^{2\pi} \sin\left(\sqrt{\rho^2\cdot\cos^2\varphi + \rho^2\cdot\sin^2\varphi}\right)\cdot\rho\,d\varphi\,d\rho$$

$$= \int_0^1\int_0^{2\pi} \rho\sin\rho\,d\varphi\,d\rho = \int_0^1 \left[\rho\sin\rho\cdot\varphi\right]_0^{2\pi} d\rho$$

$$= 2\pi\int_0^1 \rho\sin\rho\,d\rho \overset{\text{P.P.}}{=} 2\pi\left(\left[\rho\cos\rho\right]_0^1 - \int_0^1\cos\rho\,d\rho\right) = 2\pi(\cos 1 + \sin 1)$$

Posunuté pol. souřadnice

$x = \rho\cos\varphi + x_0 \qquad y = \rho\sin\varphi + y_0$



• Vypočítejte $\displaystyle\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{1-x^2}} dy\,dx = I \qquad J = \rho$

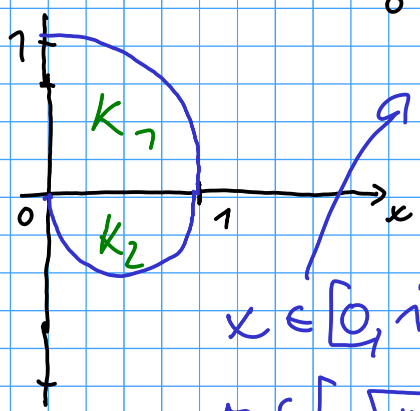
$y = \sqrt{1-x^2}$

$x^2 + y^2 = 1$

$y = -\sqrt{x-x^2}$

$y^2 + x^2 - x = 0$

$y^2 + (x - \frac{1}{2})^2 = \frac{1}{4}$



$x \in [0,1]$

$y \in [-\sqrt{x-x^2}, \sqrt{1-x^2}]$

$K_1: \ \rho \in [0,1]$, $\varphi \in [0, \pi/2]$

$K_2: \ \rho \in [0, 1/2]$, $\varphi \in [\pi, 2\pi]$

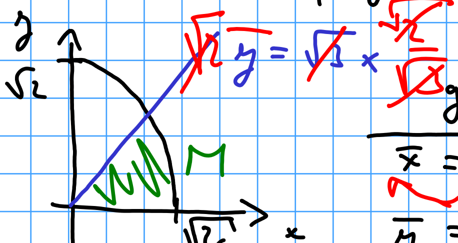
$$I = \iint_{K_1} dy\,dx + \iint_{K_2} dy\,dx$$

$$= \int_0^1 \int_0^{\pi/2} \rho \, d\varphi\, d\rho + \int_0^{1/2} \int_{\pi}^{2\pi} \rho\, d\varphi\, d\rho$$

$$= \frac{\pi}{2}\int_0^1 \rho\, d\rho + \pi \int_0^{1/2} \rho\, d\rho = \frac{\pi}{2}\left[\frac{\rho^2}{2}\right]_0^1 + \pi\left[\frac{\rho^2}{2}\right]_0^{1/2}$$

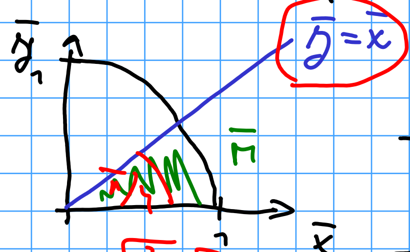
$$= \frac{\pi}{2}\cdot\frac{1}{2} + \pi\cdot\frac{1}{8} = \underline{\frac{3}{8}\pi}$$

Určete obsah části roviny ležící v 1. kvadrantu uvnitř elipsy $3x^2+y^2=2$ a pod přímkou $y=\sqrt{3}\,x$.



$\bar{x} = \sqrt{\frac{3}{2}}\, x$

$\bar{y} = \frac{1}{\sqrt{2}}\, y$



transf. do pol. s.

$\rho \in [0,1]$

$\varphi \in [0, \pi/4]$

$x = \sqrt{\frac{2}{3}}\,\bar{x}$

$y = \sqrt{2}\,\bar{y}$

$3x^2+y^2=2 \xrightarrow{x=0} y=\sqrt{2}$

$\xrightarrow{y=0} x = \sqrt{\frac{2}{3}}$

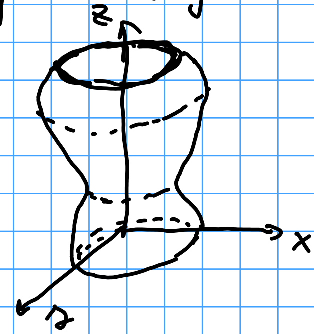
$\begin{vmatrix} \sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{2} \end{vmatrix}$

$$\iint_M dx\,dy = \iint_{\bar{M}} |J_y|\, d\bar{x}\, d\bar{y} = \int_0^1 \int_0^{\pi/4} \frac{2}{\sqrt{3}}\cdot\rho\, d\varphi\, d\rho$$

Transformace do válcových souřadnic:

$$x = \rho\cos\varphi \qquad y = \rho\sin\varphi \qquad z = z \quad \to \quad J = \rho$$

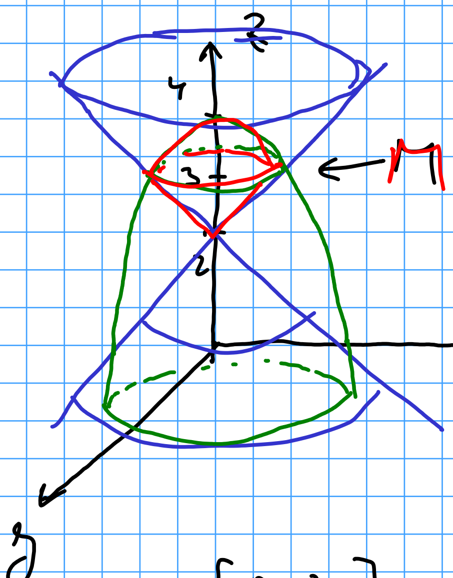
Používá se u množin, jejichž průřez je kružnice, jejíž poloměr závisí na z.



Rotační tělesa podél osy z.

• Spočítejte objem oblasti M ohraničené plochami

$$\sigma: (z-2)^2 = x^2+y^2 \qquad \sigma \cap \tau:$$
$$\tau: 4-z = x^2+y^2 \qquad (z-2)^2 = 4-z$$
$$z(z-3) = z^2-3z = 0$$



Řezy ploch rovinami $x=0$, $y=0$:

σ $\quad x=0: \quad (z-2)^2 = y^2$

$$z = \pm y + 2$$

$\quad y=0: \quad (z-2)^2 = x^2$

$$z = \pm x + 2$$

τ $\quad x=0: \quad 4-z = y^2$

$$z = 4-y^2$$

$\quad y=0: \quad z = 4-x^2$

$\varphi \in [0, 2\pi]$

$\rho \in [0, 1]$ (Pořadí si volíte)

$z \in [\rho+2, 4-\rho^2]$

$$(z-2)^2 \geq x^2+y^2 = \rho^2 \qquad z \geq \rho+2$$
$$4-z \geq x^2+y^2 = \rho^2 \qquad z \leq 4-\rho^2$$

$$\iiint_M dx\,dy = \int_0^{2\pi}\int_0^1\int_{\rho+2}^{4-\rho^2} \rho \, dz\, d\rho\, d\varphi = \int_0^{2\pi}\int_0^1 \rho(4-\rho^2-(\rho+2))\, d\rho\, d\varphi$$

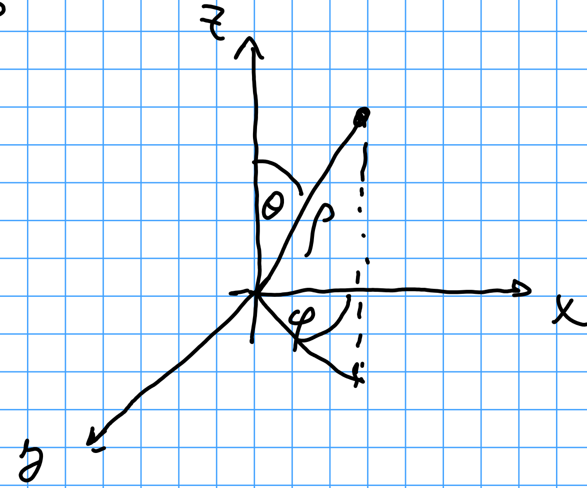
$$= \int_0^{2\pi}\int_0^1 -\rho^3-\rho^2+3\rho \, d\rho\, d\varphi = \int_0^{2\pi} \underbrace{-\frac{1}{4}-\frac{1}{3}+\frac{3}{2}}_{-3-4+3\cdot 6/12}\, d\varphi = \frac{5}{6}[\varphi]_0^{2\pi} = \frac{5}{3}\pi$$

Sférické souřadnice

$$x = \rho \cos\varphi \sin\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\theta$$



$\rho \geq 0$, $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$

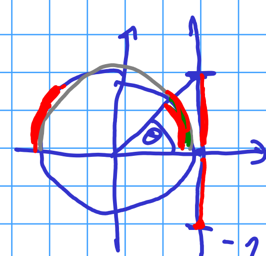
$$J = \rho^2 \sin\theta$$

- $$\iiint_M (x^2+y^2)\, z^2 \, dx\, dy\, dz, \text{ kde}$$

$$M = \{[x,y,z] \in \mathbb{R}^3;\ \underbrace{1 \leq x^2+y^2+z^2 \leq 4}_{1},\ \underbrace{x^2+y^2 \leq z^2}_{2},\ \underbrace{z \geq 0}_{3}\}$$

Převedeme do ~~pol.~~ sfér. souř.

1. $1 \leq \rho^2 \leq 4$ $\qquad 1 \leq \rho \leq 2$



2. $\rho^2 \cos^2\varphi \sin^2\theta + \rho^2 \sin^2\varphi \sin^2\theta \leq \rho^2 \cos^2\theta$

$$\rho^2 \sin^2\theta \leq \rho^2 \cos^2\theta \quad / : \rho^2$$

$$\sin^2\theta \leq \cos^2\theta$$

$$\mathrm{tg}^2\theta \leq 1 \ \rightarrow\ |\mathrm{tg}\,\theta| \leq 1$$

$$\rightarrow -1 \leq \mathrm{tg}\,\theta \leq 1 \ \rightarrow\ 0 \leq \theta \leq \frac{\pi}{4},\ \frac{3\pi}{4} \leq \theta \leq \pi$$

3. $\overset{>0}{\rho} \cos\theta \geq 0 \ \rightarrow\ \cos\theta \geq 0 \ \rightarrow\ \theta \in [0, \frac{\pi}{2}] \cup \cancel{[\frac{3}{2}\pi, 2\pi]}$

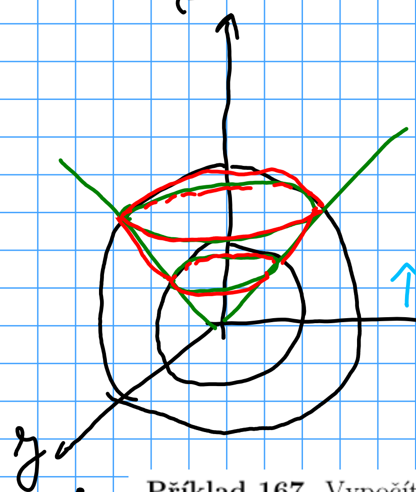
$$\iiint_M (x^2+y^2) z^2 \, dx\, dy\, dz =$$

$$\int_1^2 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} (\rho^2 \cos^2\varphi \sin^2\theta + \rho^2 \sin^2\varphi \sin^2\theta)\, \rho \cos\theta \cdot \overbrace{\rho^2 \sin\theta}^{J} \, d\theta\, d\varphi\, d\rho$$

$$= \int \int \int_0^{\pi/4} \rho^2 \sin^2\theta \cdot \rho \cos\theta \cdot \rho^2 \sin\theta \, d\theta\, d\varphi\, d\rho$$

$$= \left(\int_1^2 \rho^5 \, d\rho\right) \cdot \left(\int_0^{2\pi} d\varphi\right) \cdot \left(\int_0^{\pi/4} \sin^3\theta \cos\theta \, d\theta\right)$$

$$= \left[\frac{\rho^6}{6}\right]_1^2 \cdot \left[\varphi\right]_0^{2\pi} \cdot \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/4} = \frac{63}{6} \cdot 2\pi \cdot \frac{1}{16} = \frac{21}{16}\pi$$



$\rho \in [1, 2]$

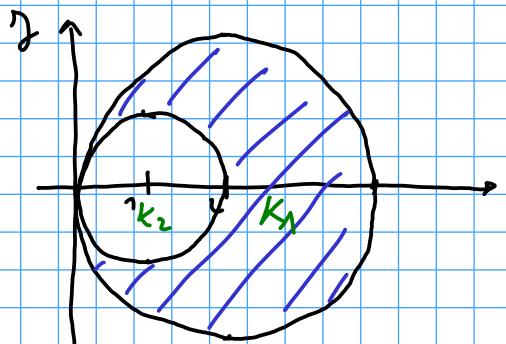
$\varphi \in [0, 2\pi]$

$\theta \in [0, \frac{\pi}{4}]$

Příklad 167. Vypočítejte ~~obsah množiny~~

$$M = \{[x,y] \in \mathbb{R}^2 : (x-1)^2 + y^2 - 1 \geq 0, (x-2)^2 + y^2 - 4 \leq 0\}.$$

$\iint_M x \, dx\, dy$.



$$\iint_M x \, dx\, dy = \iint_{K_1 - K_2} x \, dx\, dy$$

$$= \iint_{K_1} x \, dx\, dy - \iint_{K_2} x \, dx\, dy$$

$$\iint_{K_1} x \, dx\, dy = \int_0^{2\pi}\int_0^2 (\rho\cos\varphi + 2)\rho \, d\rho \, d\varphi = \int_0^{2\pi} \left[\frac{\rho^3}{3}\cos\varphi + \rho^2\right]_0^2 d\varphi$$

$$= \int_0^{2\pi} \frac{8}{3}\cos\varphi + 4 \, d\varphi = \left[\frac{8}{3}\sin\varphi + 4\varphi\right]_0^{2\pi} = 8\pi$$

$$\iint_{K_2} x \, dx\, dy = \int_0^{2\pi}\int_0^1 (\rho\cos\varphi + 1)\rho \, d\rho \, d\varphi = \int_0^{2\pi} \left[\frac{\rho^3}{3}\cos\varphi + \frac{1}{2}\rho^2\right]_0^1 d\varphi$$

$$= \int_0^{2\pi} \frac{1}{3}\cos\varphi + \frac{1}{2} \, d\varphi = \left[\frac{1}{3}\sin\varphi + \frac{1}{2}\varphi\right]_0^{2\pi} = \pi$$

$$\iint_M x \, dx\, dy = 8\pi - \pi = \underline{\underline{7\pi}}$$