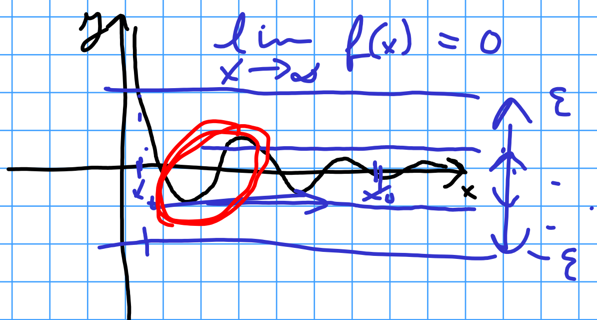
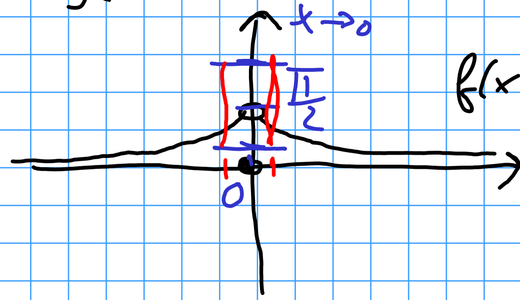
Limity

$\lim\limits_{x\to\infty} f(x) = 0$



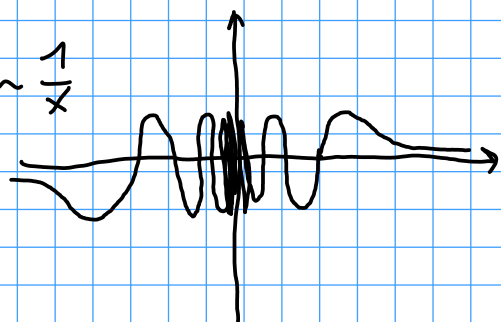
$\lim\limits_{x\to 0} f(x) = \frac{\pi}{2}$



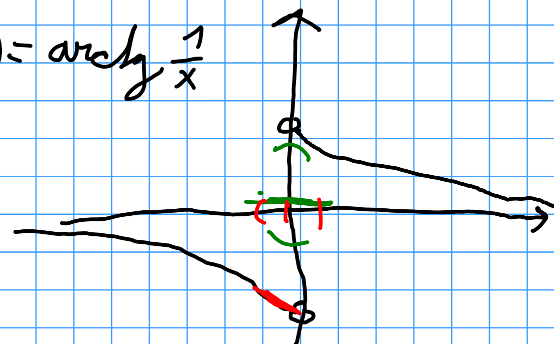
$f(x) = \begin{cases} \left|\operatorname{arctg}\frac{1}{x}\right| & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

příklady neexistence limit:

$f(x) = \sin\frac{1}{x}$



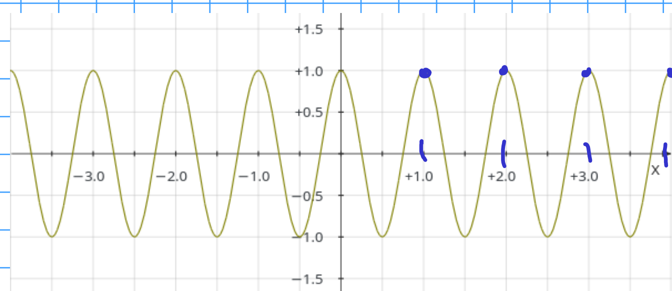
$f(x) = \operatorname{arctg}\frac{1}{x}$



limita funkce x posloupnosti

$\lim\limits_{x\to\infty} \cos(2\pi x)$ - neex.

$\lim\limits_{n\to\infty} \cos(2\pi n) = 1$



pravidla pro počítání s limitami - viz sbírka

- upravujeme tak dlouho, dokud nemůžeme „dosadit"

- $\lim\limits_{x\to\infty} \frac{1}{x^n} = \begin{cases} \left|\frac{1}{\infty}\right| = 0 & , \text{pro } n > 0 \\ \left|\frac{1}{0^+}\right| = \infty & , \text{pro } n < 0 \end{cases}$

- $\lim\limits_{x\to\infty} \frac{x^2+\sqrt{x}}{2x^2+1} = \left|\frac{\infty}{\infty}\right| = \lim\limits_{x\to\infty} \frac{\cancel{x^2}\left(1+\frac{\sqrt{x}}{x^2}\right)}{\cancel{x^2}\left(2+\frac{1}{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\left(1+\frac{\sqrt{x}}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2+\frac{1}{x^2}\right)} =$

$= \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{1}{x^{3/4}}}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x^2}} = \frac{1+0}{2+0} = \frac{1}{2}$

u limit v $+\infty$ si můžeme zapamatovat jak rychle rostou základní fce a pak jen porovnat nejrychleji rostoucí členy $\log_2(x) = \frac{\ln x}{\ln}$

$$1 \ll \log_2 x \sim \ln x \ll x \ll x^2 \ll \dots \ll 2^x \ll 3^x \ll \dots \ll x! \ll x^x$$

- $\lim\limits_{x\to\infty} \dfrac{x^3+2x+3e^x+\ln x}{x^4+1+2^x+e^x} = \lim\limits_{x\to\infty} \dfrac{\cancel{e^x}\left(\underbrace{\frac{x^3}{e^x}+\frac{2x}{e^x}}_{\to 0}+3+\underbrace{\frac{\ln x}{e^x}}_{\to 0}\right)}{\cancel{e^x}\left(\underbrace{\dots}_{\to 0}+\underbrace{\left(\frac{2}{e}\right)^x}_{\to 0}+1\right)} = 3$

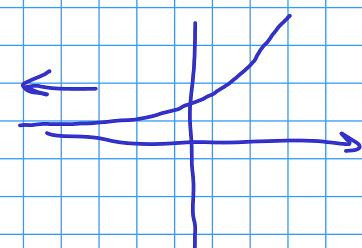
$$= \left|\lim \frac{3e^x}{e^x}\right| = 3$$

$$\sqrt[3]{x^3+2x} = \sqrt[3]{x^3\left(1+\frac{2}{x^2}\right)} = \underbrace{\sqrt[3]{x^3}}_{=x}\sqrt[3]{1+\frac{2}{x^2}}$$

- $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^3+2x}+x}{x+\sqrt[3]{x}} = 2 = \lim\limits_{x\to\infty} \dfrac{\cancel{x}\left(\sqrt[3]{1+\underbrace{\frac{2}{x^2}}_{\to 0}}+1\right)}{\cancel{x}\left(1+\underbrace{\frac{\sqrt[3]{x}}{x}}_{\to 0}\right)} = \dfrac{\sqrt[3]{1}+1}{1}$

$$x^{1/3} \ll x$$

- $\lim\limits_{x\to-\infty} \dfrac{x^4+\overbrace{e^x}^{\to 0}}{2x^{32}+2\underbrace{e^x}_{\to 0}+1} \approx \lim\limits_{x\to-\infty} \dfrac{x^4}{2x^{32}} = \lim\limits_{x\to-\infty} \dfrac{x^2}{2} = +\infty$



- $\lim\limits_{x\to\infty} \ln x - \ln(2x+3) =$

$$\lim_{x\to\infty} \ln\left(\frac{x}{2x+3}\right) = \ln\left(\lim_{x\to\infty}\frac{x}{2x+3}\right) = \ln\frac{1}{2} = -\ln 2$$

$$\ln 2^{-1} = (-1)\cdot\ln 2$$

- $\lim\limits_{x\to\infty} \left(\overbrace{\sqrt{x+2}}^{A} - \overbrace{\sqrt{x-2}}^{B}\right) \cdot \dfrac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}} = \dfrac{(x+2)-(x-2)}{\sqrt{x+2}+\sqrt{x-2}} = \lim\limits_{x\to\infty}\dfrac{4}{\sqrt{}+\sqrt{}}$

$$|\infty - \infty|$$

$$(A-B)\cdot(A+B) = A^2-B^2$$

$$= \left|\frac{4}{\infty}\right| = 0$$

$A - B$

$\cdot \lim\limits_{x\to 0} \frac{\sqrt[3]{x^2+1}-1}{x} = \lim\limits_{x\to 0} \frac{\cancel{x^2}+1-1}{\cancel{x}\left(\left(\sqrt[3]{x^2+1}\right)^2+\sqrt[3]{x^2+1}+1\right)} = \left|\frac{0}{3}\right| = 0$

$x \to 0$ · $1+1+1$

$(A-B)\cdot(A^2+AB+B^2) = A^3-B^3$

která není konstantní

viz. https://math.stackexchange.com/questions/167926/formal-basis-for-variable-substitution-in-limits

substituce – když využijeme funkce ~~spojité~~ v okolí limitního bodu, není problém

$\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\cdot \lim\limits_{n\to -\infty}\left(1+\frac{1}{n}\right)^n = \left|\begin{matrix} m=-n \\ m\to\infty\end{matrix}\right| = \lim\limits_{m\to\infty}\left(1+\frac{1}{-m}\right)^{-m}$

$= \lim\limits_{m\to\infty}\left(\frac{1}{1-\frac{1}{m}}\right)^m = \lim\limits_{m\to\infty}\left(\frac{1}{\frac{m-1}{m}}\right)^m = \lim\limits_{m\to\infty}\left(\frac{m}{m-1}\right)^m$

$= \lim\limits_{m\to\infty}\left(\frac{m-1+1}{m-1}\right)^m = \lim\limits_{m\to\infty}\left(1+\frac{1}{m-1}\right)^m$

$= \lim\limits_{m\to\infty}\left(1+\frac{1}{m-1}\right)^{m-1}\cdot\left(1+\frac{1}{m-1}\right) = e$

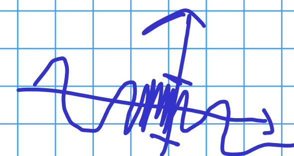
$\cdot \lim\limits_{x\to 0}\frac{\sin x^2}{x^2} = \left|\begin{matrix} y=x^2 \\ y\to 0\end{matrix}\right| = \lim\limits_{y\to 0}\frac{\sin y}{y} = 1$

$\cdot \lim\limits_{n\to\infty}\frac{(n+2)!-3(n!)}{(n+2)!+1} = \lim \frac{(n+2)(n+1)\,n!-3\,n!}{(n+2)(n+1)\,n!-1}$

$= \lim\limits_{n\to\infty}\frac{\cancel{n!}\cdot\left((n+2)(n+1)-3\right)}{\cancel{n!}\cdot\left((n+2)(n+1)-\frac{1}{n!}\right)} = \lim\limits_{n\to\infty}\frac{n^2+3n-1}{n^2+3n+2-\frac{1}{n!}}$

$= \underline{1}$

• $\lim\limits_{x\to 0} x \cdot \sin\frac{1}{x} = \lim x \cdot \cancel{\lim \sin \frac{1}{x}}$



$= |0 \cdot \underline{\text{ohr}}| = 0$

$(1^{-\infty})$

e^x je spojitá

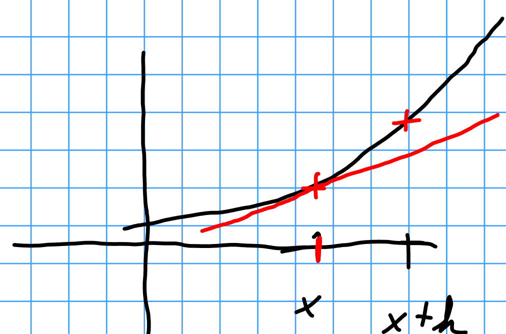
• $\lim\limits_{x\to 0^+} x^{\ln x} = \lim\limits_{x\to 0^+} e^{\ln(x^{\ln x})} = e^{\underbrace{\lim\limits_{x\to 0^+} \ln(x^{\ln x})}} = |e^{\infty}| = \underline{\underline{\infty}}$

$\lim\limits_{x\to 0^+} \ln(x^{\ln x}) = \lim\limits_{x\to 0^+} \ln x \cdot \ln x = (-\infty)\cdot(-\infty) = \infty$

• $\lim\limits_{x\to 0} \sqrt{\frac{\sin x}{x}} = \sqrt{\lim\limits_{x\to 0}\frac{\sin x}{x}} = \sqrt{1} = 1$

Derivace

• $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$



Derivace základních funkcí

$(x^0)' = (1)' = (c)' = 0$ | $(\ln x)' = \frac{1}{x}$ | $(\sin x)' = \cos x$

$x' = 1$ | $(e^x)' = e^x$ | $(\cos x)' = -\sin x$

$(x^n)' = n x^{n-1}$, $n \neq 0$ | $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ | $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

Pravidla pro počítání s limitami:

$$(f(x)+g(x))' = f'(x)+g'(x) \quad \Big| \quad (f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2} \quad \Big| \quad \big(f(g(x))\big)' = f'(g(x))\cdot g'(x)$$

$$f(f^{-1}(x)) = x \xrightarrow{\text{derivace}} f'(f^{-1}(x))\cdot(f^{-1})'(x) = 1 \Rightarrow (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

- z def. $(\sin x)' = \lim\limits_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x \underbrace{\frac{\cos h - 1}{h}}_{\to 0} + \cos x \underbrace{\frac{\sin h}{h}}_{\to 1} = \underline{\underline{\cos x}}$$

$$\lim_{h\to 0}\frac{\cos h - 1}{h}\cdot\frac{\cos h+1}{\cos h+1} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h+1)} = \lim_{h\to 0} \underbrace{\frac{-\sin h}{h}}_{\to -1}\cdot\frac{\overbrace{\sin h}^{\to 0}}{\underbrace{\cos h}_{\to 1}+1} = 0$$

$$\sin^2 h + \cos^2 h = 1 \Rightarrow \cos^2 h - 1 = -\sin^2 h = -\sin h\cdot \sin h$$

- $f(x) = \operatorname{arctg}(x^2+1)$ $\qquad \big(f(g(x))\big)' = f'(g(x))\cdot g'(x)$

$$\operatorname{arctg}' = \frac{1}{x^2+1}$$

$$f'(x) = \frac{1}{(x^2+1)^2+1}\cdot\big(\underbrace{(x^2)'}_{2x} + \underbrace{1'}_{0}\big) = \frac{2x}{(x^2+1)^2+1}$$

- $f(x) = \dfrac{1}{\sqrt[3]{x+1}} = \dfrac{1}{(x+1)^{1/3}} = (x+1)^{-1/3}$

$$f'(x) = \left(-\frac{1}{3}\right)\cdot(x+1)^{-\frac{1}{3}-1}\cdot(1+0) = \left(-\frac{1}{3}\right)\cdot(x+1)^{-\frac{4}{3}} = \frac{-1}{3\sqrt[3]{(x+1)^4}}$$

$$\cdot\ (tg\ x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x + \sin x\ (+\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \boxed{\frac{1}{\cos^2 x}} \quad \text{zapamatovat}$$

$$1 + tg^2(x)$$

$$\cdot\ f(x) = e^{\overbrace{\sin(x^2)\cdot \ln x}^{*}}$$

$$f'(x) = e^{(*)} \cdot \left(\overbrace{(\cos(x^2)\cdot 2x)}^{(\sin(x^2))'} \cdot \ln x + \sin x^2 \cdot \frac{1}{x} \right)$$

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1} \quad \cdot\ (e^x)' = \lim_{h\to 0} \frac{\overset{e^x\cdot e^h}{e^{x+h}} - e^x}{h} = \lim_{h\to 0} e^x \cdot \underbrace{\frac{(e^h - 1)}{h}}_{1}$$

$$= e^x$$

$$f(x) = e^x \quad f^{-1}(x) = \ln x \qquad (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$