Derivace implicitní fce

$y = \pm\sqrt{1-x^2}$

graf implicitní fce : body splňující rovnici např. $\boxed{x^2+y^2=1}$
↳ funkce $y = y(x)$ splňující tuto rovnost
např. $\boxed{y = \pm\sqrt{1-x^2}}$ splňuje

$x^2 + y(x)^2 = 1$

im. fce derivujeme tak, že si y představíme jako složenou funkci $y(x)$

- $y \cdot \sin x + x \cdot \cos y = 1$, určete 2. der. v bodě $[x,y] = [1,0]$

$$y(x) \cdot \sin x + x \cdot \cos y(x) = 1 \qquad \Big/ \frac{d}{dx}$$

$$\Rightarrow \underline{y'(x) \cdot \sin x} + y(x) \cdot \cos x + \cos y(x) + \underline{x \cdot (-\sin y(x)) \cdot y'(x)} = 0$$

$$y' \cdot (\sin x - x \sin y) = -y \cos x - \cos y$$

$$\rightarrow y' = \frac{-y\cos x - \cos y}{\sin x - x \sin y} = \frac{\overset{1}{\cos \overset{0}{y}} \cdot \overset{0}{y} \cos \overset{1}{x}}{\overset{1}{x} \sin \underset{0}{y} - \sin \underset{1}{x}}$$

$$\underline{y'' \cdot \sin x} + y' \cdot \cos x + y' \cos x + y \cdot (-\sin x) + (-\sin y)\, y' +$$
$$- \left(\sin y \cdot y' + x \cos y \cdot (y')^2 + \underline{x \sin y \cdot y''}\right) = 0$$

$$y'' (\sin x - x \sin y) = \sin y \cdot y' + x \cos y\, (y')^2 - 2 y' \cos x$$
$$+ \sin x \cdot y + \sin y\, y'$$

$$y'' = \frac{\cancel{s(\overset{0}{y}) \cdot y'} + \overset{1}{x} \cdot c(\overset{0}{y}) (y')^2 - 2 y' c(\overset{1}{x}) + \cancel{s(\overset{1}{x}) \cdot \overset{0}{y}} + \cancel{s(\overset{0}{y}) \cdot y}}{s(\underset{1}{x}) - \cancel{\underset{1}{x} \cdot s(\underset{0}{y})}}$$

$$y'(x,y) \rightarrow y'(1,0) = \frac{1}{-\sin 1}$$

$$= \frac{(y')^2 - 2 y' \cos 1}{\sin 1} = \frac{\frac{1}{\sin^3 1} + 2 \frac{1}{\sin^2 1} \cos 1}{\cancel{\sin 1}}$$

←

Průběh funkce - pokračování

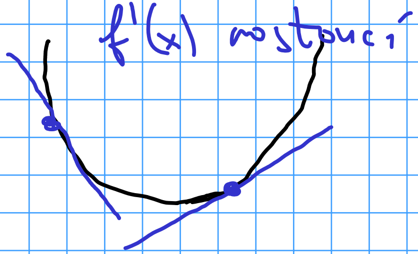
vyšetřování průběhu funkce celkově - viz sbírka

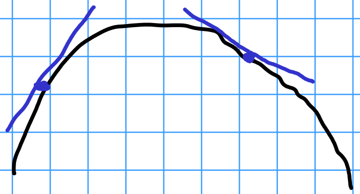
minule: intervaly monotonie (rostoucí, kles.)

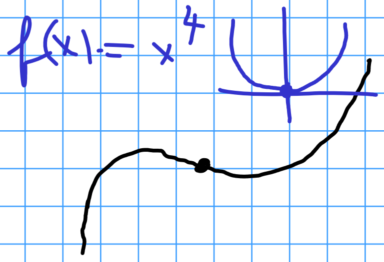
stac. body $\longrightarrow$ lok. extrémy

- konvexnost, konkávnost, inflexní bod

$f''(x) > 0$ $\quad$ $f''(x) < 0$ $\quad$ $f''(x) = 0$







• určete, kde je funkce $\underline{f(x) = \frac{x}{1+x^2}}$ konvexní, konkávní a určete inflexní body

$$f'(x) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{\underline{(1+x^2)^2}}$$

$$f''(x) = \frac{-2x\cdot(1+x^2)^{\cancel{2}} - (1-x^2)\cdot 2\cancel{(1+x^2)}\cdot 2x}{(1+x^2)^{\cancel{4}\,3}} = \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3}$$

$$= \frac{2x^3 - 6x}{(\underbrace{\boxed{1+x^2}}_{>0})^3} \substack{>\\=\\<} 0 \iff 2x^3 - 6x \substack{<\\=\\>} 0$$

$$2x(x^2 - 3) = 0$$

nulové body: $x = 0, -\sqrt{3}, \sqrt{3}$

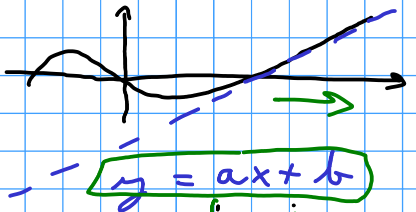
f'' ⊖ ^{0} ⊕ ^{0} ⊖ ^{0} ⊕

∩ $-\sqrt{3}$ ∪ 0 ∩ ^{1} $\sqrt{3}$ ^{2} ∪

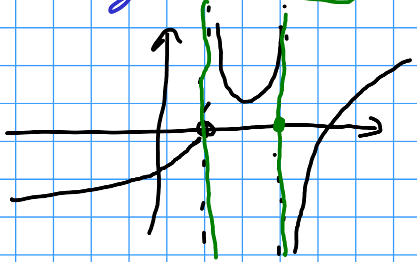
konvexní: $(-\sqrt{3}, 0), (\sqrt{3}, \infty)$

konkávní: $(-\infty, -\sqrt{3}), (0, \sqrt{3})$

$\to\infty$: $f(x) \approx ax+b \to \lim\limits_{x\to\infty} \frac{f(x)}{x} \approx a + \underbrace{\frac{b}{x}}_{\to 0}$



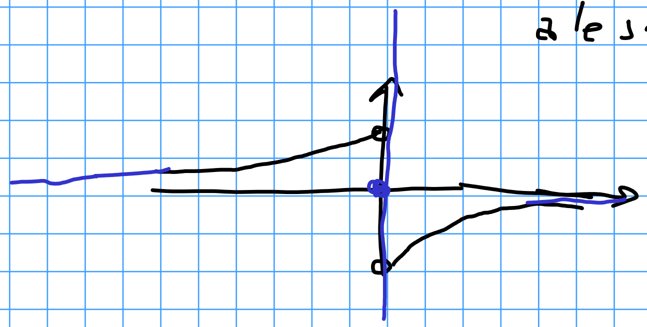
- asymptoty $<$ se směrnicí / bez směrnice



- se směrnicí : $y = ax+b$

$$a = \lim_{\substack{x\to\infty\\(-\infty)}} \frac{f(x)}{x} \qquad b = \lim_{\substack{x\to\infty\\(-\infty)}} \left(f(x) - ax\right)$$

- bez směrnice : bod nespojitosti, kde to utíká alespoň z jedné strany do $\pm\infty$



$\to$ as. se směr. ? $+\infty$ $y=0$, $-\infty$ $y=0$

as. bez směr. ? Není

- $f(x) = x \cdot e^{\frac{1}{x-1}}$ $\boxed{\text{určete}}$ asymptoty $D(f) = \mathbb{R} \setminus \{1\}$

bez směrnice : $x = 1$ $\lim\limits_{x\to 1^-} x e^{\frac{1}{x-1}} = \left|1 \cdot e^{\frac{1}{1^- - 1}} = 1 \cdot e^{\frac{1}{0^-}}\right|$

$= \left|e^{-\infty}\right| = 0 \neq \pm\infty$ ✗

$\lim\limits_{x\to 1^+} x\, e^{\frac{1}{x-1}} = \left|1 \cdot e^{\frac{1}{1^+ - 1}} = e^{\frac{1}{0^+}} = e^{\infty}\right| = \infty$ ✓

$\Rightarrow$ v bodě $x = 1$ je as. bez sm.

se směrnicí : $a = \lim\limits_{\substack{x\to\infty\\-\infty}} \frac{x e^{1/x-1}}{x} = \lim\limits_{\substack{x\to\infty\\-\infty}} e^{\frac{1}{x-1}}$

$= e^{\lim\limits_{\substack{x\to\infty\\-\infty}} \frac{1}{x-1}} = e^0 = 1$

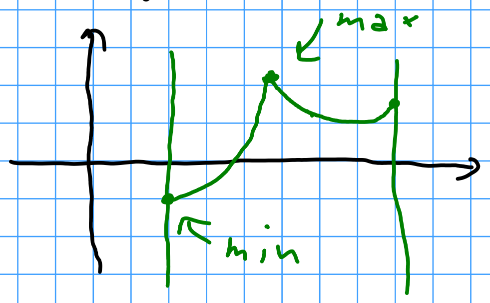
$b = \lim\limits_{\substack{x\to\infty\\-\infty}} x e^{\frac{1}{x-1}} - \overset{1}{a} x = |\text{typ } \infty - \infty|$

$= \lim x \left(e^{\frac{1}{x-1}} - 1\right) = \lim\limits_{\substack{x\to\infty\\-\infty}} \frac{e^{\frac{1}{x-1}} - 1}{\frac{1}{x}}$

$$\stackrel{l'H}{=} \lim \frac{e^{\frac{1}{x-1}} \cdot \frac{-1}{(x-1)^2}}{-\frac{1}{x^2}} = \lim_{\substack{x \to \infty \\ -\infty}} \underbrace{e^{\frac{1}{x-1}}}_{\to 1} \cdot \overbrace{\frac{x^2}{x^2-2x+1}}^{\to 1} = 1$$

as. v $\pm\infty$: $\underline{y = x + 1}$

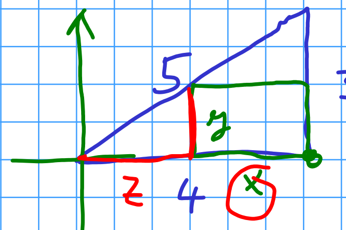
Optimalizace (globální extrémy)



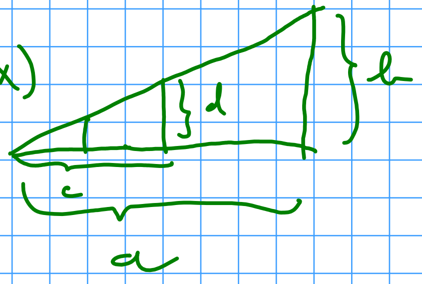
globální extrém se může nacházet v:

- stacionárním bodě
- bodě nespojitosti
- krajním bodě

- Vložte do trojúhelníku se stranami 3 cm, 4 cm, 5 cm co největší obdélník (s co největším obsahem).



$y = \frac{3}{4}(4-x)$



$$\frac{a}{b} = \frac{c}{d}$$

$z : y \to 4 : 3$

$$\frac{4-x}{y} = \frac{4}{3} \Rightarrow y = \underline{\frac{3}{4}(4-x)}$$

$$S = y \cdot x = \left(3 - \frac{3}{4}x\right)x = 3x - \frac{3}{4}x^2 \qquad x \in [0,4]$$

$$S' = 3 - \frac{3}{2}x = 0 \qquad \text{stac. b.: } x_0 = 2$$

$S(0) = S(4) = 0$

$S(2) = 3 \rightarrow \max$

$x_0 = 2$

$y_0 = 3 - \frac{3}{4} \cdot 2 = \frac{3}{2}$

$\frac{y_0}{x_0} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$

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celkový průběh fce (doplnění)

$$f(x) = \frac{x}{1+x^2}$$

$$f(x) \quad \begin{array}{ccc} - & 0 & + \\ \hline & 0 & \end{array}$$

$$f'(x) = \frac{1 \cdot (1+x^2) - x \cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{\underbrace{(1+x^2)^2}_{>0}}$$

$$f'(x) \quad \begin{array}{ccccc} \ominus & 0 & \oplus & 0 & \ominus \\ \hline \searrow & -1 & \nearrow & 1 & \searrow \\ & \min & & \max & \end{array}$$

$$f''(x) = \frac{-2x \cdot (1+x^2)^{\cancel{2}} - (1-x^2) \cdot 2\cancel{(1+x^2)} \cdot 2x}{(1+x^2)^{\cancel{4}\,3}} = \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3}$$

$$= \frac{2x^3 - 6x}{(\underbrace{1+x^2}_{>0})^3} \quad \overset{>}{\underset{<}{=}} 0 \iff 2x^3 - 6x \overset{<}{\underset{>}{=}} 0$$

$$2x(x^2 - 3) = 0$$

nulové body: $x = 0, -\sqrt{3}, \sqrt{3}$

$$f'' \quad \begin{array}{ccccccc} \ominus & 0 & \oplus & 0 & \ominus & 0 & \oplus \\ \hline \cap & -\sqrt{3} & \cup & 0 & \cap & \sqrt{3} & \cup \end{array}$$

konvexní: $(-\sqrt{3}, 0), (\sqrt{3}, \infty)$

konkávní: $(-\infty, -\sqrt{3}), (0, \sqrt{3})$

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim \frac{\frac{x}{1+x^2}}{x} = \lim \frac{1}{1+x^2} = 0$$

$$b = \lim_{x \to \infty} f(x) - \overset{0}{a} \cdot x = \lim_{x \to \infty} \frac{x}{1+x^2} = 0$$

$y = 0$ v $\pm\infty$

$f \quad - \quad \overset{0}{\underset{0}{|}} \quad +$

$f' \quad - \quad \underset{-1}{|} \quad + \quad \underset{1}{|} \quad -$

$\searrow \quad -1 \quad \nearrow \quad 1 \quad \searrow$

$f'' \quad - \quad \underset{-\sqrt{5}}{|} \quad + \quad \underset{0}{|} \quad - \quad \underset{\sqrt{5}}{|} \quad +$

$\cap \quad -\sqrt{5} \quad \cup \quad 0 \quad \cap \quad \sqrt{5} \quad \cup$