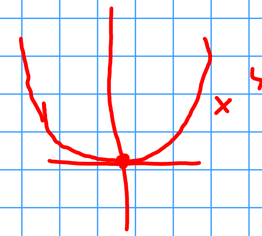
inflexní bod – změna znaménka $f''(x) \neq 0$



Diferenciál

v bodě x_0: $df(x) = f'(x_0)\,dx$ $\quad dx = x - x_0$

$$f(x) - f(x_0) \approx f'(x_0)\overbrace{(x-x_0)}^{=dx} = df(x)$$

$$f(x) \approx f(x_0) + df(x) = f(x_0) + f'(x_0)(x-x_0)$$

↑ s rovností zadává rovnici tečny v bodě x_0

Taylorův polynom

stupně n v bodě x_0:

$$f(x) \approx T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$f(x) = T_n(x) + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \qquad c \in [x_0, x]$$

- Nalezněte Taylorův pol. stupně k funkce $f(x) = \frac{1}{x}$ v bodě $x_0 = 1$.

$$f(x_0) = \frac{1}{x_0} = \frac{1}{1} = 1$$

$$f'(x) = \left(\frac{1}{x}\right)' = \frac{-1}{x^2} \longrightarrow f'(1) = -1$$

$$f''(x) = (-x^{-2})' = 2x^{-3} \longrightarrow f''(1) = 2$$

$$f'''(x) = (2x^{-3})' = -6x^{-4} \to f'''(1) = -6$$

$$\vdots$$

$$f^{(k)}(x) = (x^{-1})^{(k)} = (-1)\cdot(x^{-2})^{(k-1)} = (-1)(-2)(x^{-3})^{(k-2)}$$

$$= \dots = (-1)(-2)\cdot\dots\cdot(-k)\quad x^{-(k-1)}$$

$$= (-1)^k \cdot k! \cdot \frac{1}{x^{k-1}}$$

$$\to f^{(k)}(1) = k!\cdot(-1)^k \to \frac{f^{(k)}(1)}{k!} = (-1)^k$$

$$T_k(x) = f(1) + f'(1)\cdot(x-1) + \frac{1}{2!}f''(1)\,(x-1)^2 +$$
$$+ \dots + \frac{1}{k!}\,f^{(k)}\,(x-1)^k$$

$$= 1 - 1\cdot(x-1) + 1\cdot(x-1)^2 - \dots + (-1)^k\,(x-1)^k$$

<u>pozn.</u> platí $\quad \frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n \quad$ pro $\quad x \in (0,2)$

(součet geom. řady) $\qquad (x-1)^2 = (1-x)^2$

$$\frac{1}{1-(1-x)} = \frac{1}{x}$$

• Jakého stupně potřebujeme vzít Taylorův polynom funkce $f(x) = x \ln x$ v bodě $x_0 = 1$, abychom jím aproximovali hodnotu funkce $f(2) = 2\ln 2$ chybou $\leq \frac{1}{10}$?

$$f(x) = T_n(x) + R_n(x)$$

$$f(1) = 0, \quad f'(x) = (x \ln x)' = \ln x + x \cdot \frac{1}{x}$$

$= 1 \cdot \ln 1$

$$= \ln x + 1 \quad \rightarrow \quad f'(1) = 1$$

$$f''(x) = \frac{1}{x} \quad \rightarrow \quad f''(1) = 1$$

$$(-1)^k = (-1)^{k-2} \cdot \underbrace{(-1)^2}_{=1}$$

$$f^{(k)}(x) = \left(\frac{1}{x}\right)^{(k-2)} = (-1)^{k-2}(k-2)! \quad \frac{1}{x^{k-3}}$$

$$\rightarrow \quad f^{(k)}(1) = (-1)^k (k-2)! \qquad \text{pro } k \geq 2$$

$$T_n(x) = f(1) + f'(1)(x-1) + \frac{1}{2!} f''(1)(x-1)^2$$

$$+ \ldots + \frac{1}{n!} f^{(n)}(1)(x-1)^n$$

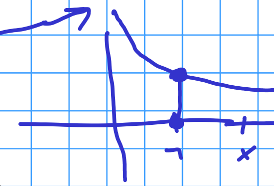
$$= 0 + (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \ldots + \frac{(-1)^n (n-2)!}{n!} \cdot (x-1)^n$$

$$= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3 \cdot 2}(x-1)^3 + \frac{1}{4 \cdot 3}(x-1)^4 - \frac{1}{5 \cdot 4}(x-1)^5 + \ldots$$

$$+ \frac{(-1)^n}{n \cdot (n-1)}(x-1)^n$$

$$|R_n(x)| = \left| \frac{(-1)^{n+1}(n-1)!}{(n+1)!} \cdot \frac{1}{c^{k-1}} \cdot (x-1)^{n+1} \right| \quad \text{, kde } c \in [1, x]$$

klesající na (pro $x > 1$)



$$|R_n(2)| \leq \frac{1}{(n+1)\,n} \cdot (2-1)^{n+1} = \frac{1}{(n+1)\cdot n} \leq \frac{1}{12} \overset{\text{chceme}}{\leq} \frac{1}{10}$$

Platí pro $n = 3$. $\quad f(2) \approx 1{,}389$

$$T_3(2) \approx 1{,}335$$

Pozn. z tvaru R_n lze vidět, že pro $x > 2$ se chyba bude zvětšovat se stupněm n.

Primitivní funkce

$F(x)$ primitivní k $f(x) \iff F'(x) = f(x)$

když $F(x)$ je prim. k $f(x)$, pak i $F(x) + c$

neurčitý integrál: $\int f(x)\,dx = \{F(x) + c \mid c \in \mathbb{R}\}$

Základní Pravidla: $\int c \cdot f(x)\,dx = c \int f(x)\,dx$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Integrály elementárních funkcí: (podle derivací)

$$(x^{k+1})' = (k+1)\,x^k \Rightarrow \int x^k\,dx = \frac{x^{k+1}}{k+1} + c$$

$$\left(\frac{x^{k+1}}{k+1}\right)' = x^k \qquad \text{pro } k \neq -1$$

$\int \sin x \, dx = -\cos x + c$ (pozor na znaménko)

$\int \cos x \, dx = \sin x + c$

$\int \frac{1}{x} \, dx = \ln x + c$ $\int \frac{1}{x^2+1} \, dx = \operatorname{arctg} x$

$(\ln(f(x)))' = \frac{1}{f(x)} \cdot f'(x) \Rightarrow \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)|$

$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$ $\int e^x \, dx = e^x + c$

viz sbírka

$c \in \mathbb{R}$

- $\int x^3 + \frac{1}{2}x + 2e^x - \sin x \, dx = \frac{x^4}{4} + \frac{x^2}{4} + 2e^x + \cos x + c$
- $\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + c = \frac{2\sqrt{x^3}}{3}$
- $\int \frac{3x+1}{x^2+1} \, dx = \int \frac{3x}{x^2+1} + \frac{1}{x^2+1} \, dx = \int \frac{3}{2} \frac{2x}{x^2+1} + \frac{1}{x^2+1} \, dx$

$(x^2+1)' = 2x$ $= \frac{3}{2} \ln|x^2+1| + \operatorname{arctg} x$

Pravidlo per-partes

$(u(x) \cdot v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

$\int u(x) \, v'(x) \, dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x) \, dx$

- $\int x \ln x \, dx = \left| \begin{matrix} u = \ln x & u' = \frac{1}{x} \\ v' = x & v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$

$= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$

- $\int x e^x \, dx$

- $\int x^2 \sin x \, dx$

- $\int 4x \operatorname{arctg} x \, dx$