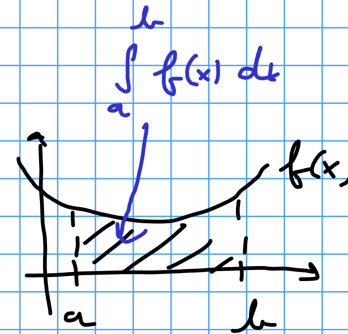
Určitý integrál



$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

- $$\int_1^2 x^2+\sqrt{x}\,dx = \left[\frac{x^3}{3} + \frac{x^{3/2}}{3/2}\right]_1^2 = \frac{8}{3} + \frac{2\sqrt{8}}{3} - \left(\frac{1}{3} + \frac{2}{3}\right) = \frac{5+4\sqrt{2}}{3}$$

per-partes

$$\int_a^b u(x)\cdot v'(x)\,dx = \left[u(x)\cdot v(x)\right]_a^b - \int_a^b u'(x)\cdot v(x)\,dx$$

substituce

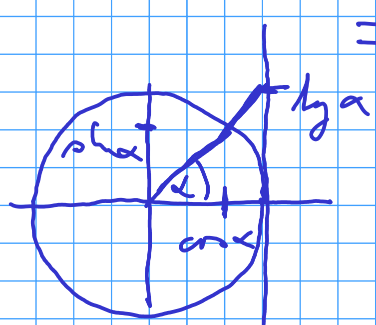
$$\int_a^b f(\varphi(x))\,\varphi'(x)\,dx = \left|\begin{array}{l} t = \varphi(x) \\ dt = \varphi'(x)\,dx \\ a \to \varphi(a) \\ b \to \varphi(b) \end{array}\right| = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$$

$$(\sqrt{x})^2 = |x| = x$$

- $$\int_0^1 \sqrt{x}\,\mathrm{arctg}(\sqrt{x})\,dx = \left|\begin{array}{ll} u = \mathrm{arctg}(\sqrt{x}) & u' = \frac{1}{x+1}\cdot\frac{-1}{2\sqrt{x}} \\ v' = \sqrt{x} & v = \frac{2\sqrt{x^3}}{3} \end{array}\right|$$

$$= \left[\frac{2\sqrt{x^3}}{3}\,\mathrm{arctg}\sqrt{x}\right]_0^1 - \int_0^1 \frac{\cancel{2}\,x\cancel{\sqrt{x}}}{3}\cdot\frac{(-1)}{\cancel{2\sqrt{x}}}\cdot\frac{1}{x+1}\,dx$$

$$= \left(\frac{2}{3}\overbrace{\mathrm{arctg}\sqrt{1}}^{=\pi/4} - 0\right) - \frac{1}{3}\int_0^1 \underbrace{\frac{x+1-1}{x+1}}_{1-\frac{1}{x+1}}\,dx$$



$$= \frac{\pi}{6} - \frac{1}{3}\left[x - \ln|x+1|\right]_0^1$$

$$= \frac{\pi}{6} - \frac{1}{3}\left((1-\ln 2) - (0-0)\right) = \boxed{\frac{\pi-2}{6} + \frac{\ln 2}{3}}$$

$$\bullet \int_0^1 \sqrt{x}\,\operatorname{arctg}(\sqrt{x})\,dx = \left| \begin{array}{l} t=\sqrt{x} \quad x=t^2 \\ dt = \frac{1}{2\sqrt{x}}\,dx \\ 1 \to \sqrt{1} = 1 \\ 0 \to \sqrt{0} = 0 \end{array} \right|$$

$$= \int_0^1 \frac{1}{2\sqrt{x}}\, 2\underbrace{x}_{t^2}\, \operatorname{arctg}(\underbrace{\sqrt{x}}_{t})\,\underbrace{dx}_{dt} = \int_0^1 2t^2 \operatorname{arctg}(t)\,dt$$

$$\bullet \int_{-a}^{a} x\sqrt{a^2-x^2}\,dx = \left| \begin{array}{l} x = a\sin t \\ dx = a\cos t\,dt \\ t = \arcsin\frac{x}{a} \\ -a \to \arcsin -1 = -\frac{\pi}{2} \\ a \to \frac{\pi}{2} \end{array} \right|$$

0

$$= \int_{-\pi/2}^{\pi/2} a\sin t \cdot \sqrt{a^2 - a^2\sin^2 t}\cdot a\cos t\,dt$$

0

$$= \int_{-\pi/2}^{\pi/2} a^3 \sin t \sqrt{\underbrace{1-\sin^2 t}_{=\cos^2 t}} \cdot \cos t\,dt$$

0

$$= \int_{-\pi/2}^{\pi/2} a^3 \underbrace{\sin t\cos^2 t\,dt}_{-ds} \quad \left| \begin{array}{l} s = \cos t \\ ds = -\sin t\,dt \\ \frac{\pi}{2} \to 0 \\ -\frac{\pi}{2} \to 0 \\ 0 \to 1 \end{array} \right.$$

0

$|\sin t\cos^2 t$ je lichá fce$|$

$$= 0 \qquad -\int_1^0 a^3 s^2\,ds = a^3\left[\frac{s^3}{3}\right]_0^1 = \frac{a^3}{3}$$

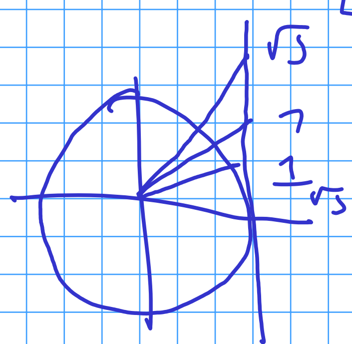
$$\bullet \int_0^1 \frac{e^x}{e^{2x}+3}\,dx \quad \left| \begin{array}{l} t = e^x \\ dt = e^x\,dx \\ 1 \to e^1 = e \\ 0 \to e^0 = 1 \end{array} \right| = \int_1^e \frac{1}{t^2+3}\,dt$$

$$= \frac{1}{3}\int_1^e \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\,dt = \frac{1}{3}\left[\frac{1}{\frac{1}{\sqrt{3}}}\operatorname{arctg}\left(\frac{t}{\sqrt{3}}\right)\right]_1^e$$

$$= \frac{1}{\sqrt{3}}\left(\operatorname{arctg}\left(\frac{e}{\sqrt{3}}\right) - \underbrace{\operatorname{arctg}(1)}_{\pi/2}\right)$$

$$\bullet \int_{\pi/6}^{\pi/3} \operatorname{tg}^2 x\,dx = \int_{\pi/6}^{\pi/3} \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x} - 1\,dx$$

$$= \Big[\operatorname{tg} x - x\Big]_{\pi/6}^{\pi/3} = \left(\sqrt{3} - \frac{\pi}{3}\right) - \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$$



$$= \frac{3\sqrt{3}-3}{3} - \frac{\pi}{6}$$

$$\left(\left| \begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t \\ dx = \frac{1}{t^2+1}\,dt \end{array} \right| = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{t^2}{t^2+1}\,dt \right)$$

- $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \Big[\,2\sqrt{x}\,\Big]_0^1 = 2 - \lim_{x\to 0} 2\sqrt{x} = 2$$

$$\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \big[\ln x\big]_0^1 = 0 - \lim_{x\to 0^+} \ln x = \infty & \alpha = 1 \\ \left[\dfrac{x^{-\alpha+1}}{-\alpha+1}\right]_0^1 = \dfrac{1}{-\alpha+1} - \lim_{x\to 0^+} \dfrac{x^{-\alpha+1}}{-\alpha+1} = \end{cases}$$

- $$\int_0^1 \frac{-1}{x^\alpha}\,dx \quad \left|\begin{array}{l} t = \frac{1}{x} \\ dt = -2\,\frac{1}{x^2}\,dx \\ 0^+ \to \infty \\ 1 \to 1 \end{array}\right| = \frac{-1}{2}\int_\infty^1 t^{\alpha-2}\,dt$$

- $$\int_e^\infty \frac{(\ln x)^2}{x}\,dx \quad \left|\begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\,dx \\ \infty \to \lim_{x\to\infty} \ln x = \infty \\ e \to \ln e = 1 \end{array}\right| = \int_1^\infty t^2\,dt$$

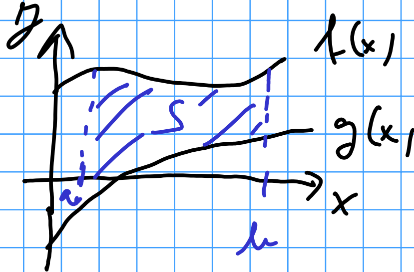
$$= \left[\frac{t^3}{3}\right]_1^\infty = \lim_{t\to\infty} \frac{t^3}{3} - \frac{1}{3} = \infty$$

diverguje

Aplikace urč. integrálu

Obsah plochy mezi křivkami

$$S = \int_a^b |f(x) - g(x)|\,dx$$

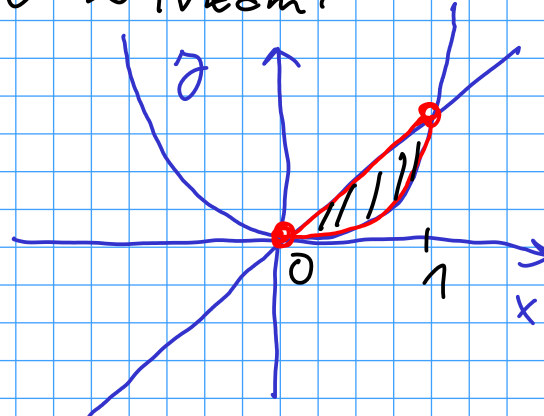


• Určete obsah plochy ohraničené křivkami

$f(x) = x$, $g(x) = x^2$

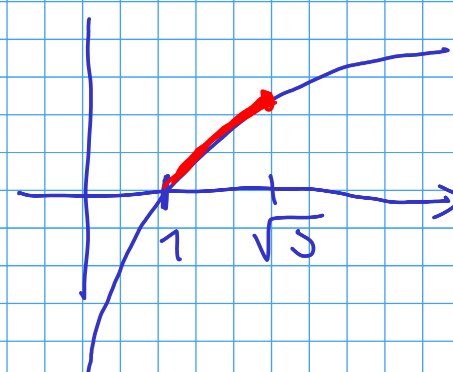
$$\int_0^1 x - x^2\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \underline{\underline{\frac{1}{6}}}$$



• Určete délku křivky $f(x) = \ln x$ na intervalu $[1, \sqrt{3}]$.

$$l = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$



$$l = \int_1^{\sqrt{3}} \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx = \underbrace{\int_1^{\sqrt{3}} \frac{1}{x}\sqrt{x^2+1}\,dx}_{dt}$$

$$= \left| \begin{array}{l} t = x^2 + 1 \\ dt = 2x\,dx \\ 1 \to 2 \\ \sqrt{3} \to 4 \end{array} \right| = \frac{1}{2x}\int^{\sqrt{3}} \frac{2x\,\sqrt{x^2+1}\,dx}{x^2+1-1} = \frac{1}{2}\int_2^4$$

$$= \frac{1}{2}\int_2^4 \frac{\sqrt{t}}{t-1} \cdot \frac{2\sqrt{t}}{2\sqrt{t}}\,dt \quad \left| \begin{array}{l} s = \sqrt{t} \\ ds = \frac{1}{2\sqrt{t}}\,dt \\ 2 \to \sqrt{2},\ 4 \to 2 \end{array} \right.$$

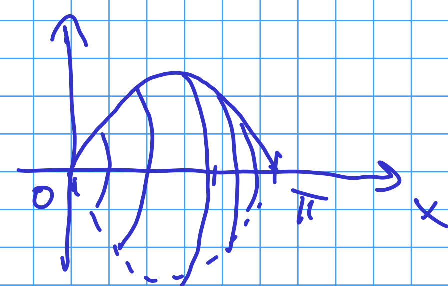
$$= \int_{\sqrt{2}}^{2} \frac{s^2-1+1}{s^2-1}\, ds = \int_{\sqrt{2}}^{2} 1 + \frac{1}{s^2-1}\, ds$$

$\rightarrow$ parc. zl. $= \frac{1/2}{s-1} - \frac{1/2}{s+1}$

$$= \left[s\right]_{\sqrt{2}}^{2} + \frac{1}{2}\left[\ln|s-1| - \ln|s+1|\right]_{\sqrt{2}}^{2} =$$ stačí dosadit

- Určete objem rotačního tělesa vzniklého rotací křivky $f(x) = \sin x$ kolem osy x na int. $[0, \pi]$.

$$V = \pi \int_a^b f^2(x)\, dx$$



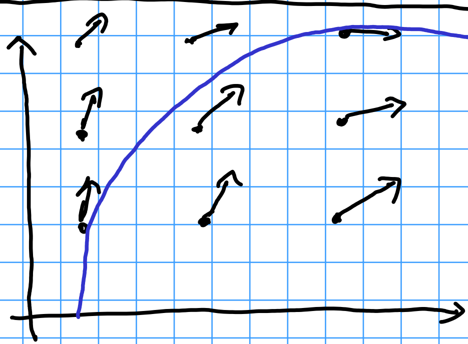
$$V = \pi \int_0^{\pi} \sin^2 x\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} 1 - \cos 2x\, dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left((\pi - 0) - (0 - 0)\right) = \frac{\pi^2}{2}$$

Diferenciální rovnice

$$y' = f(x, y)$$



Rovnice se separovatelnými proměnnými

$$\frac{dy}{dx} = y' = f(x) \cdot g(y) \rightarrow \frac{dy}{dx} = f(x) \cdot g(y)$$

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

- $(1+e^x)\, y\, y' = e^x \quad , \quad y(0) = 1$

$$y \frac{dy}{dx} = \frac{e^x}{1+e^x} \longrightarrow \int y\, dy = \int \frac{e^x}{1+e^x}\, dx$$

$$\frac{y^2}{2} = \ln|1+e^x| + C \qquad \text{obecné řešení}$$

$$y(0) = 1 : \quad \frac{1}{2} = \ln|1+e^0| + C = \ln 2 + C$$

$$C = \frac{1}{2} - \ln 2$$

Řešení splňující podm.: $\frac{y^2}{2} = \ln|1+e^x| + \frac{1}{2} - \ln 2$

$$y' = f\left(\frac{y}{x}\right) \rightarrow u = \frac{y}{x} \qquad xu = y$$

$$u + xu' = y'$$

$$\rightarrow u + xu' = f(u) \ \dots \text{ sep. pr.}$$

- $xy' + y \ln x = y \ln y$