Part I

Basics of coding theory and linear codes

CODING and CRYPTOGRAPHY

IV054

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In spite of the fact that both coding and cryptography areas have already many very efficient systems using only very small memories, new and new applications require to develop again and again new, faster, and less memory demanding systems for both coding and cryptography.

IV054-CRYPTO team;

Teaching stuff

- Prof. Jozef Gruska DrSc - lecturer
- RNDr. Matej Pivoluska PhD - tutorials end CRYPTO team member
- RNDr Lukáš Boháč - head of CRYPTO-team
- Mgr. Libor Cáha PhD, member of CRYPTO-team
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Prerequisites: Basics of discrete mathematics and linear algebra See: "Appendix" in http://www.fi.muni.cz/usr/gruska/crypto21,

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Homeworks 5-6 sets of homeworks of 6-8 exercises designed and evaluated by our CRYPTO-team created mainly from some of best former IV054-students

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Each student will get 5 questions. Number of questions a student has to respond will depend on the number of points received for homeworks. Each student will get automatically A in case (s)he received number of points from exercises $>=$ 85% of MAX - maximal number of points any student got from exercises.

Automatically a student gets B, with an easy way to get A, in case the number of points (s)he received is in interval (75,85)% of MAX.......

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- Detailed slides of all lectures. (Each chapter will consists of a (i) short prologue, (ii) basic materials and an (iii) Appendix -for much demanding students.

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- Appendix of fundamental discrete math and linear algebra - 45 pages
- Two lecture notes of solved examples (at least 100 in each one) and short (2-3) pages overviews for all chapters.
- Posted solutions of homeworks

IV054 - goals

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1. To learn beautiful and powerful **basics** of the coding theory and of the classical as well as quantum modern cryptography and steganography-watermarking needed for all informaticians; in almost all areas of informatics and for transmission and storing information.

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2. To verify, for ambitious students, their capability to work hard to be successful in very competitive informatics+mathematics environments.

BIBLIOGRAPHY

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Fundamentals of quantum information transmission and Cryptography. Surprising and even shocking practical applications of quantum information transmission and cryptography. Top current cryptosystem for applications.
Comment: Concerning both lectures and homeworks the overall requirement for students will be significantly smaller than in previous years.

Codes basics and linear codes

Basics of coding theory
and
an introduction to linear codes

PROLOGUE - I.

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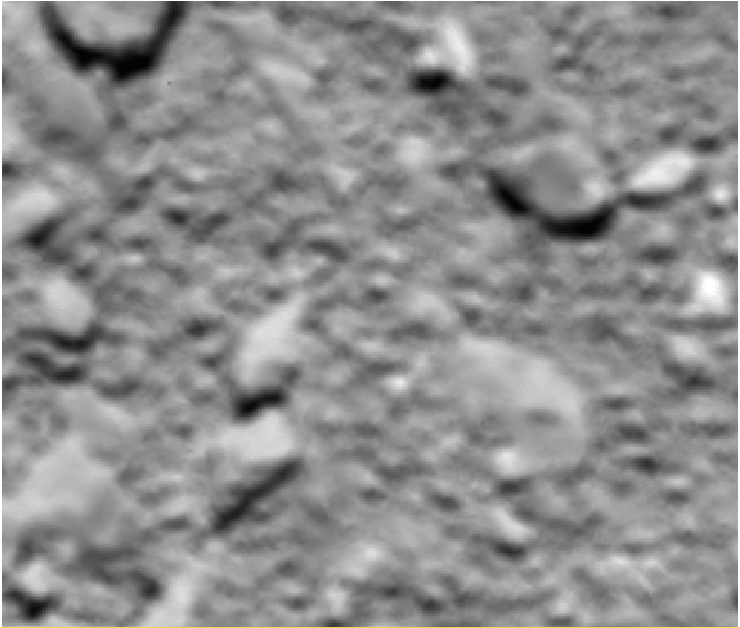
- Since that time coding theory has made another enormous progress that has allowed, among other things, almost perfect mobile communication and transmission of music in time and space.

ROSETTA spacecraft

ROSETTA LANDING - VIEW from 21 km -29.9.2016

ROSETTA LANDING - VIEW from 51 m -29.9.2016



CHAPTER 1: BASICS of CODING THEORY

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All real communication systems that work with digitally represented data, as CD players, TV, fax machines, internet, satellites, mobiles, require to use error correcting codes because all real channels are, to some extent, noisy – due to various interference/destruction caused by the environment

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Goals of coding theory are to develop systems and methods that allow to detect/correct errors caused when information is transmitted through **noisy channels**.

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- Coding theory problems are therefore among the very basic and most frequent

PROLOGUE - II.

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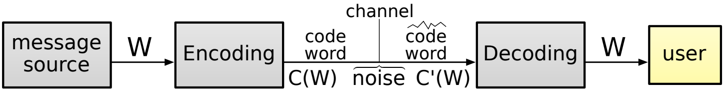
using mainly classical, but also quantum tools.

Error-correcting codes are used to correct messages when they are (erroneously) transmitted through noisy channels.



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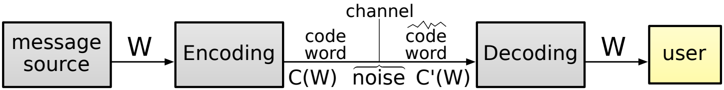

Error correcting framework

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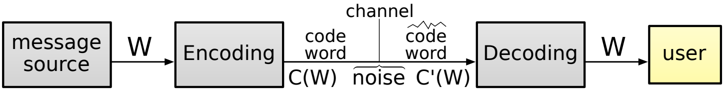
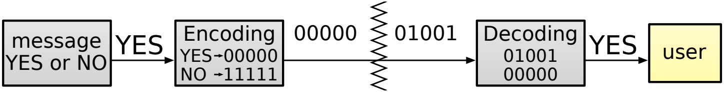


Error correcting framework

Example



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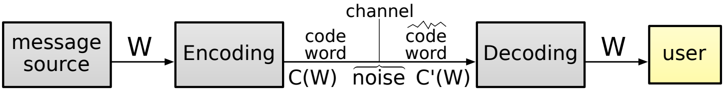

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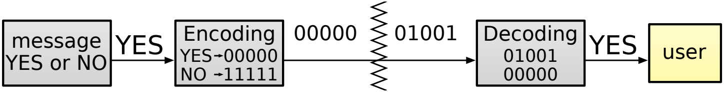

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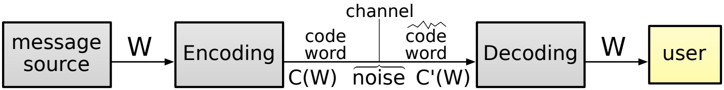

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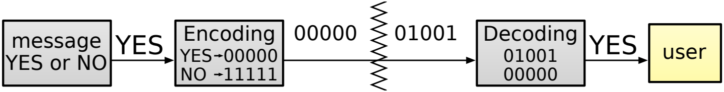

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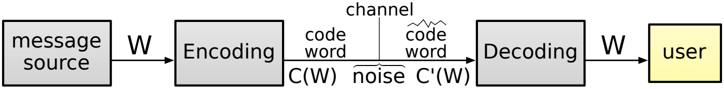

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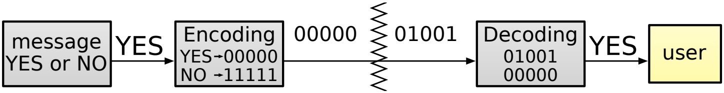

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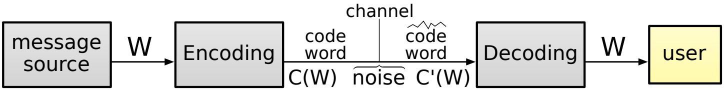

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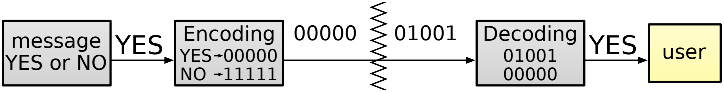

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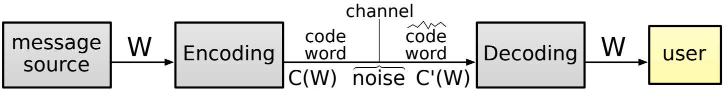

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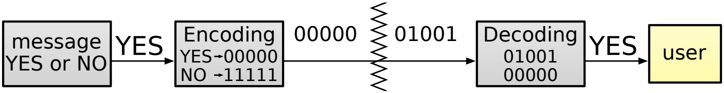

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BASIC METHOD OF FIGHTING ERRORS: REDUNDANCY!!!

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5. Corection of errors introduced in the channel should be reasonably easy.
6. As large as possible amount of information should be transferred reliably per a time unit.

BASIC METHOD OF FIGHTING ERRORS: REDUNDANCY!!!

Example: 0 is encoded as 00000 and 1 is encoded as 11111.

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- **White noise Gaussian channel that models errors in the deep space.**

BASIC CHANNEL CODING PROBLEMS

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Also many non-cryptography applications require error-correcting codes. For example, mobiles, CD-players,. . .

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- Another field of applications of error-correcting codes is that of mass memories: computer hard drives, CD-Rooms, DVDs and so on.

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The key idea is that in order to protect a message against a noise, we should encode the message by adding some redundant informationto the message.

This should be done in such a way that even if the message is corrupted by a noise, there will be enough redundancy in the encoded message to recover, or to decode the message completely.

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then the probability of an erroneous decoding (for the case of 2 or 3 errors) is

$$3p^2(1-p) + p^3 = 3p^2 - 2p^3 < p$$

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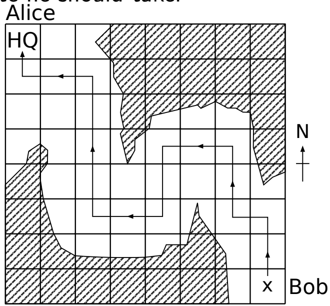

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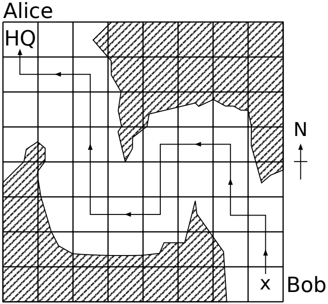

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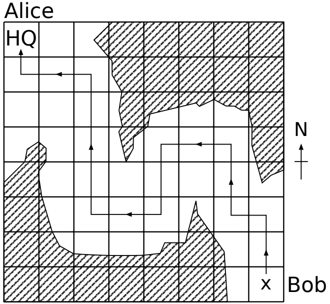

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Example: $h(10101, 01100) = 3$, $h(fourth, eighth) = 4$

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An important parameter of codes C is their minimal distance.

$$h(C) = min\{h(x, y) \,|\, x, y \in C, x \neq y\},$$

Therefore, $h(C)$ is the smallest number of errors that can change one codeword into another.

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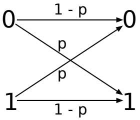

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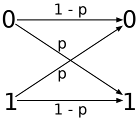

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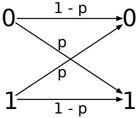

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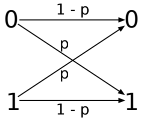

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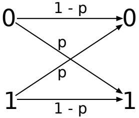

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Corollary One undetected error occurs only once every 2000 days! ($2000 \approx \frac{10^9}{5.5 \times 86400}$).

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Comment: A good (n, M, d)-code has small n, large M and also large d.

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Decoding was quite simple.

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Code rate (6/32 for Hadamard code), is an important parameter for real implementations, because it shows what fraction of the communication bandwidth is being used to transmit actual data.

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The ISBN code was designed to detect: (a) any single error (b) any double error created by a transposition

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$$(1)\ \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{Bmatrix}$$

$$(2)\ \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{Bmatrix} \begin{Bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{Bmatrix}$$

Lemma Any q-ary (n, M, d)-code over an alphabet $\{0, 1, \ldots, q-1\}$ is equivalent to an (n, M, d)-code which contains the all-zero codeword $00\ldots0$.

Proof Trivial.

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Proof Let u be a fixed word in F_q^n. The number of words that differ from u in m positions is

$$\binom{n}{m}(q-1)^m.$$

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In general, does it has a sense to look for such codes that some important sum of any two codewords is again a codeword?

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Each base **B** of C is usually represented by a (k, n) matrix, $G_{\mathbf{B}}$, so called a **generator matrix of C**, the i-th row of which is the i-th codeword of **B**.

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$C_2 = \{000, 011, 101, 110\}$ – YES

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Theorem A binary linear code of dimension k has

$$\frac{1}{k!} \prod_{i=0}^{k-1} (2^k - 2^i)$$

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Proof Operations (a) - (c) just replace one basis by another. Last two operations convert a generator matrix to one of an equivalent code.

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Theorem Let G be a generator matrix of an $[n, k]$-code. Rows of G are then linearly independent .By operations (a) - (e) the matrix G can be transformed into the form: $[I_k|A]$ where I_k is the $k \times k$ identity matrix, and A is a $k \times (n-k)$ matrix.

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APPENDIX II.

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Basic idea: Encode more (less) probable outputs of X by shorter (longer) binary words.
Example (Moorse code - 1838)

a .-	b -...	c -.-.	d -..	e .	f ..-.	g –.
h	i ..	j .—	k -.-	l .-..	m –	n -.
o —	p .–.	q –.-	r .-.	s ...	t -	u ..-
v ...-	w .–	x -..-	y -.–	z –..		

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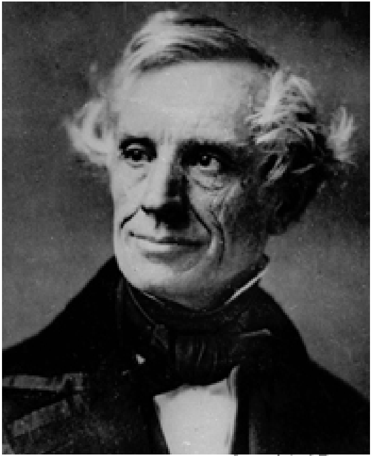

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mess.	code	mess.	code	mess.	code	mess.	code
0000	10	0100	010	1000	011	1100	11101
0001	000	0101	11001	1001	11011	1101	111110
0010	001	0110	11010	1010	11100	1110	111101
0011	11000	0111	1111000	1011	111111	1111	1111001

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Observe that this is a **prefix code** - no codeword is a prefix of another codeword.

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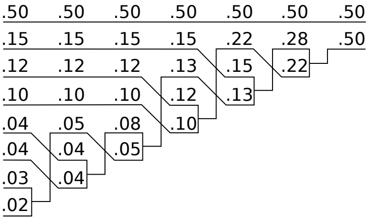

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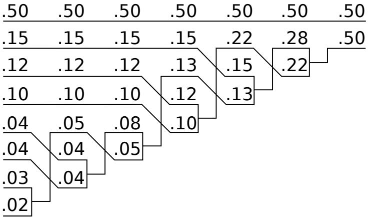

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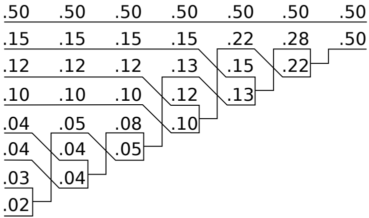

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.15	.15	.15	.15	.22	.28	.50
.12	.12	.12	.13	.15	.22	
.10	.10	.10	.12	.13		
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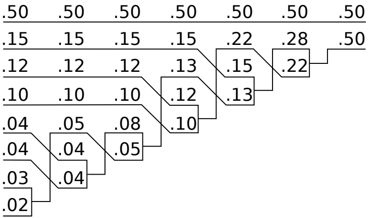

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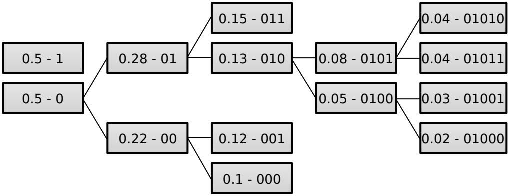

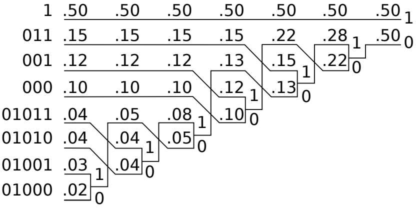

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Originally, information theory was a part of electrical engineering. Nowadays, it is an important part of mathematics and also of informatics.

APPENDIX II- ENTROPY - basics - I.

The concept of **ENTROPY** is one of the most basic and important in modern science, especially in physics, mathematics and information theory.

So called **physical entropy** is a measure of the unavailable energy in a closed thermodynamics system (that is usually considered to be a measure of the system's disorder).

Entropy of an object is a measure of the amount of energy in the object which is unable to do some work.

Entropy is also a measure of the number of possible arrangements of the atoms a system can have.

Information entropy

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On the other side, if we have a process (random variable) Y producing value 0 with probability $\frac{1}{4}$ and value 1 with probability $\frac{3}{4}$, then we are more certain that the next value of the process will be 1 than 0.

History Rudolf Clausius coined the term **entropy** in 1865.

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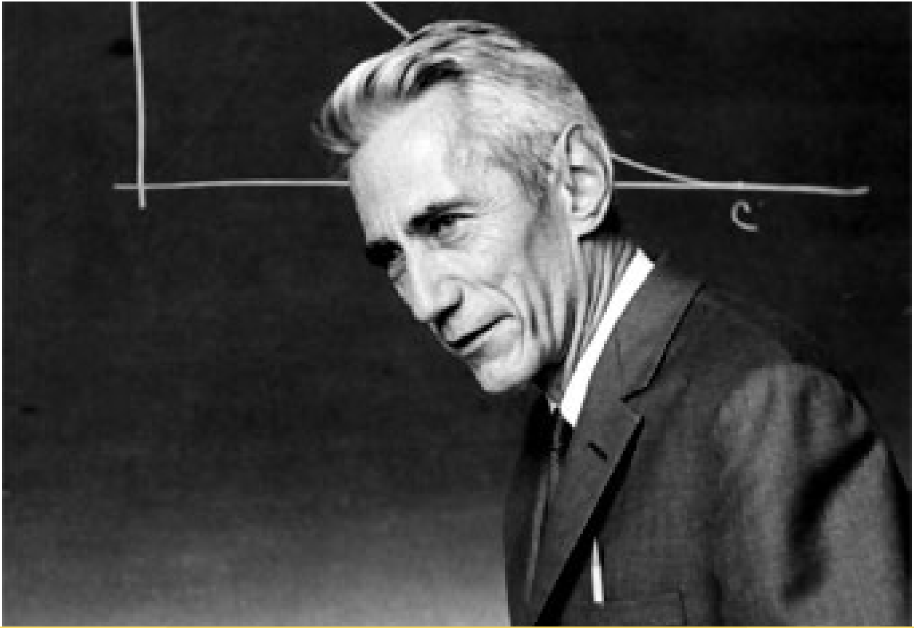
The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.



APPENDIX III. SOFT DECODING

APPENDIX

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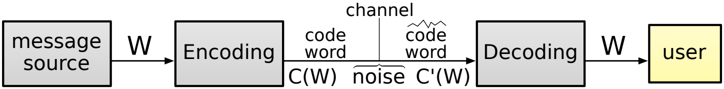

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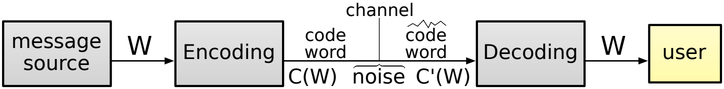

In that process a binary message is at first encoded into a binary codeword, then transmitted through a noisy channel, and, finally, the decoder receives, for decoding, a potentially erroneous binary message and makes an error correction.

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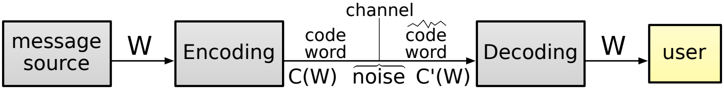

In that process a binary message is at first encoded into a binary codeword, then transmitted through a noisy channel, and, finally, the decoder receives, for decoding, a potentially erroneous binary message and makes an error correction.

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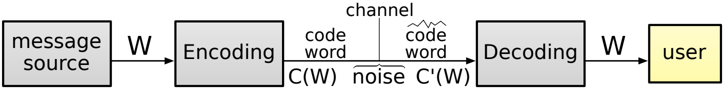

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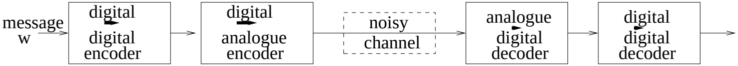

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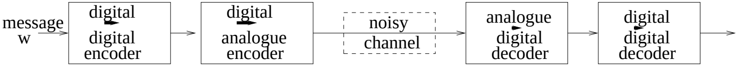

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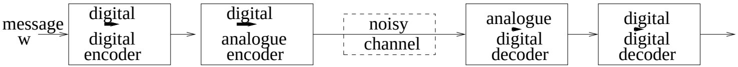

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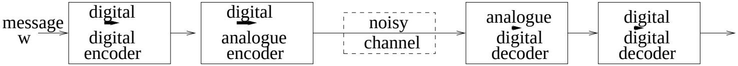

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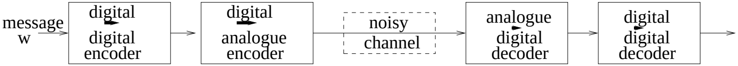

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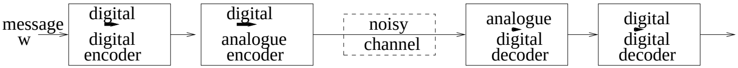

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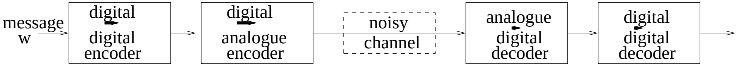

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In case the output of analogous-digital decoding is a pair (p_b, b) where p_b is the probability that the output is the bit b (or a weight of such a binary output (often given by a number from an interval $(-V_{max}, V_{max})$), we talk about a **soft decoding**.

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In order to deal with such a more general model of transmission and soft decoding, it is common to use, instead of the binary symbols 0 and 1 so-called **antipodal binary symbols** $+1$ and -1 that are represented electronically by voltage $+1$ and -1.

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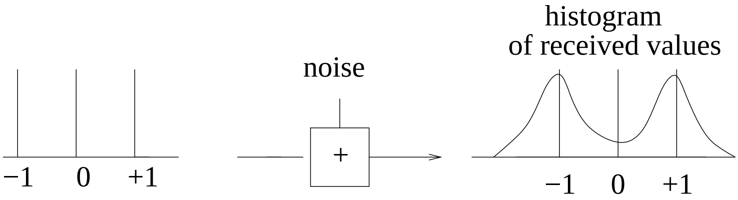

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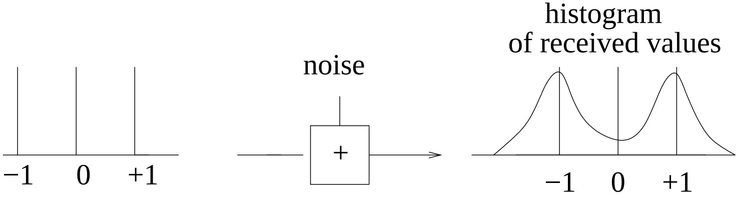

A very important case in practise, especially for space communication, is so-called additive white Gaussian noise (AWN) and the channel with such a noise is called Gaussian channel.

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For example, in an important practical case of the Gaussian white noise one search at the minimal likelihood decoding for a codeword with minimal Euclidean distance.

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Hard decoding is used mainly for block codes and soft one for stream codes. However, distinctions between these two families of codes are tending to blur.

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The binary elements 0 and 1 were first called bits by J. W. Tickle in 1943.

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For details about 13-digit ISBN see

`https://en.wikipedia.org/wiki/International_Standard_Book_Number`