Part I

Linear, Cyclic, stream and channel codes. Speccial decodings

LINEAR CODES II. - continuation, decoding

APPENDIX I.

WISDOM

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Many practically important linear codes have also an efficient decoding.

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Comment. To design linear codes we will use Galois fields $GF(q)$ with q being a prime. One can also use Galois fields $GF(q^k)$, $k > 1$, but their structure and operations are defined in a more complex way, see the Appendix.

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Encoding (code) is called systematic if for any $m \in M \subset \Sigma^*$

$$e(m) = mc_m \quad \text{for some} \quad c_m \in \Sigma^*$$

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Each base **B** of C is usually reperesented by a (k, n) matrix, $G_{\mathbf{B}}$, so called a **generator matrix of C**, the i-th row of which is the i-th codeword of **B**.

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Proof Operations (a) - (c) just replace one basis by another. Last two operations convert a generator matrix to one of an equivalent code.

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Theorem Let G be a generator matrix of an $[n, k]$-code. Rows of G are then linearly independent .By operations (a) - (e) the matrix G can be transformed into the form: $[I_k|A]$ where I_k is the $k \times k$ identity matrix, and A is a $k \times (n-k)$ matrix.

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$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow$$

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SYNDROMES APPROACH to DECODING

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Properties of binary Hamming codes Coset leaders are precisely words of weight ≤ 1. The syndrome of the word $0 \ldots 010 \ldots 0$ with 1 in j-th position and 0 otherwise is the transpose of the j-th column of H.

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Hamming codes were originally used to deal with errors in long-distance telephon calls.

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G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbols of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code.

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Rows of the last 11 columns are cyclic permutations of the first row which has 1 at those positions that are squares modulo 11, that is

$$0, 1, 3, 4, 5, 9.$$

REED-MULLER CODES

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This is an infinite, recursively defined, family of so called $RM_{r,m}$ binary linear $[2^m, k, 2^{m-r}]$-codes with

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- Matrix Q_r is obtained by considering all combinations of r rows of $G_{1,m}$ and by obtaining products of these rows/vectors, component by component. The result of each of such a multiplication constitues a row of Q_r.

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Plotkin bound implies that q-nary error-correcting codes with $d \geq n(1 - 1/q)$ have only polynomially many codewords and hence are not very interesting.

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An important example of MDS-codes are q-ary Reed-Solomon codes RSC(k, q), for $k \leq q$.

They are codes a generator matrix of which has rows labelled by polynomials X^i, $0 \leq i \leq k-1$, columns labeled by elements $0, 1, \ldots, q-1$ and the element in the row labelled by a polynomial p and in the column labelled by an element u is $p(u)$.

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Reed-Solomon codes are used in digital television, satellite communication, wireless communication, barcodes, compact discs, DVD,... They are very good to correct burst errors - such as ones caused by solar energy.

APPENDIX - I.

LDPC (Low-Density Parity Check) - CODES

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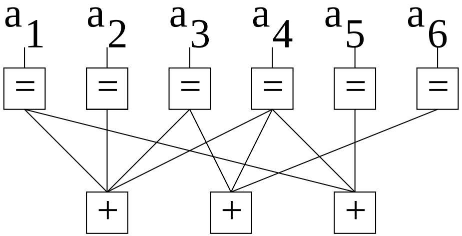
An $[n, k]$ LDPC code can be represented by a bipartite graph between a set of n top "variable-nodes (v-nodes)" and a set of bottom $(n - k)$ "parity check nodes (pc-nodes)".
Variable nodes:



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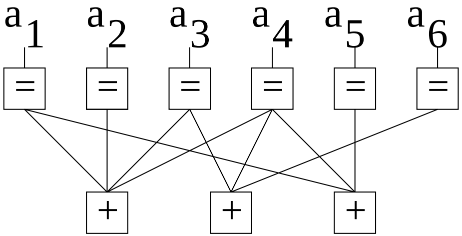

Parity check nodes:

The corresponding parity check matrix has $n-k$ rows and n columns and i-th column has 1 in the j-th row exactly in case if i-th v-node is connected to j-th c-node.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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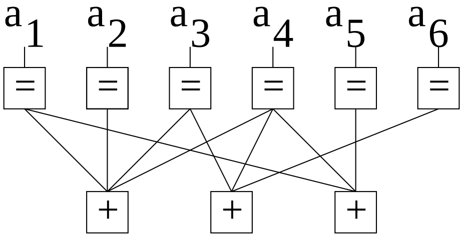
The LDPC-code with the Tanner bipartite graph for $(6,3)$ LDPC-code.



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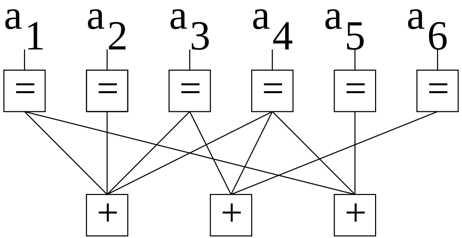

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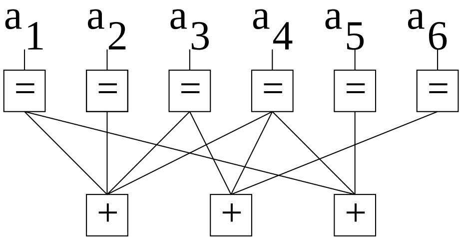

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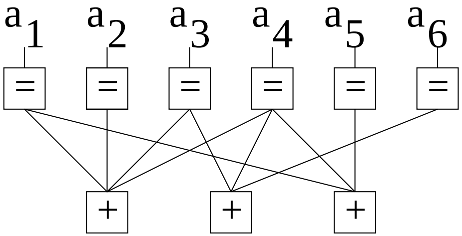

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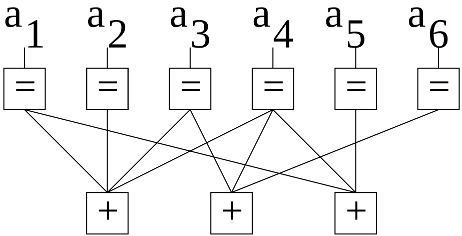

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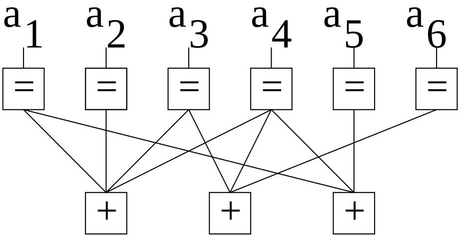

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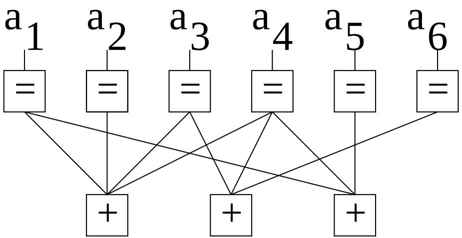

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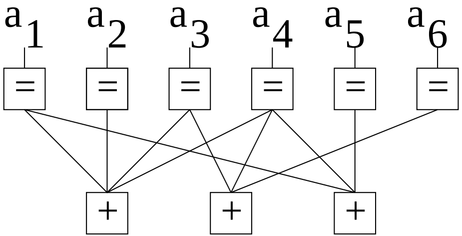

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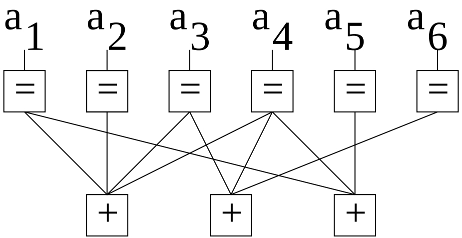

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Using so called **iterative belief propagation techniques**, LDPC codes can be decoded in time linear to their block length.

DESIGN of LDPC codes

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- Some good LDPC codes were designed through randomly chosen parity check matrices.
- Some LDPC codes are based on Reed-Solomon codes, such as the RS-LDPC code used in the 10-gigabit Ethernet standard.

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- LDPC is also used for 10Gbase-T Ethernet, which sends data at 10 gigabits per second over twisted-pair cables.
- Since 2009 LDPC codes are also part of of the Wi-Fi 802.11 standard as an optional part of 802.11n, in the High Throughput PHY specification.

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Each Reed-Muller code $RM(r,m)$ is the code of k codewords of length $n = 2^m$, to encode k messages, and distance 2^{m-r}, where

$$k = \sum_{s=0}^{r} \binom{d}{s}.$$

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They are very useful especially in those applications where one can expect that errors occur in bursts - such as ones caused by solar energy.

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For Reed-Solomon codes there is list decoding up to $1 - \sqrt{2R}$ errors.

CHANNELS (STREAMS) CODING

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A breakthrough came when D. Forney, in his PhD thesis in 1972, showed that so called concatenated codes could be used to achieve exponentially decreasing error probabilities at all data rates less than the Shannon channel capacity, with decoding complexity increasing only polynomially with the code length.

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The code rate express the amount of redundancy in the code - the lower is the code rate, the more redundancy is in the codewords.

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CHANNEL CAPACITY

Channel capacity of a communication channel, is the tightest upper bound on the (code) rate of information that can be reliably transmitted over that channel.

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By the **noisy-channel Shannon coding theorem**, the channel capacity of a given channel is the limiting code rate (in units of information per unit time) that can be achieved with arbitrary small error probability.

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The channel capacity is then defined by

$$C = \sup_{P_X(x)} I(X,Y)$$

where

$$I(X,Y) = \sum_{y \in Y} \sum_{x \in X} P_{X,Y}(x,y) \log \left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} \right)$$

is the mutual distribution - a measure of variables mutual distribution.

SHANNON NOISY CHANEL THEOREM

For every discrete memoryless channel, the channel capacity

$$C = \sup_{P_X} I(X, Y)$$

has the following properties:
1. For every $\varepsilon > 0$ and $R < C$, for large enough N there exists a code of length N and code rate R and a decoding algorithm, such that the maximal probability of the block error is $\leq \varepsilon$.
2. If a probability of the block error p_b is acceptable, code rates up to $R(p_b)$ are achievable, where and $H_2(p_b)$ is the binary entropy function. and $H_2(p_b)$ is the binary entropy function.
3. For any p_b code rates greater than $R(p_b)$ are not achievable.

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An (n,k) convolution code with a k x n generator matrix G can be used to encode a k-tuple of **message-polynomials** (polynomial input information)

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EXAMPLE 2 – when the code CC_2 is used:

$$(x^2 + x, x^3 + 1) \cdot G_2 = (x^2 + x, x^3 + 1) \cdot \begin{pmatrix} 1 + x & 0 & x + 1 \\ 0 & 1 & x \end{pmatrix}$$

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$$C_0(x) = C_{00} + C_{01}x + \ldots = (x^2 + 1)I(x)$$

and

$$C_1(x) = C_{10} + C_{11}x + \ldots = (x^2 + x + 1)I(x).$$

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An input stream $I(x) = (I_0(x), I_1(x), I_2(x), \ldots)$ is mapped into the output stream $C = (C_{00}, C_{10}, C_{01}, C_{11} \ldots)$ defined by

$$C_0(x) = C_{00} + C_{01}x + \ldots = (x^2 + 1)I(x)$$

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The first multiplication can be done by the first shift register from the next figure; second multiplication can be performed by the second shift register on the next slide and it holds

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That is the output streams C_0 and C_1 are obtained by convoluting the input stream with polynomials of G_1.

ENCODING

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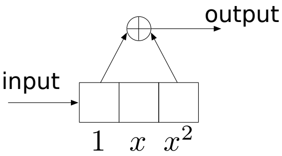

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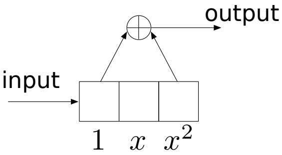

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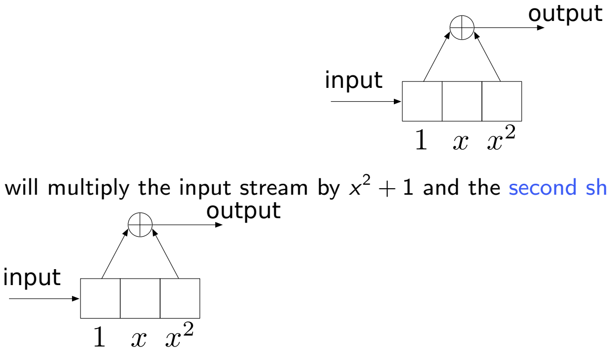

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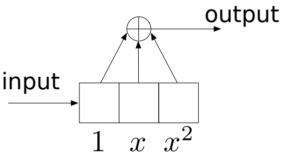

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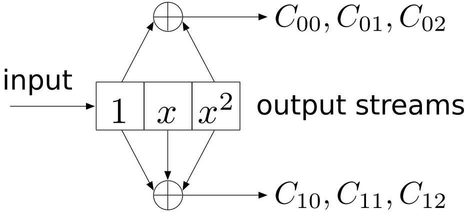

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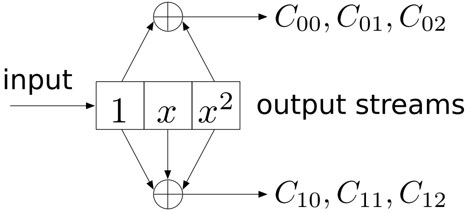

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Given $(x, y) \in \{-1, 1\} \times R$, the noise $y - x$ is distributed according to the Gaussian distribution of zero mean and standard derivation σ of the channel

$$Pr(y|x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

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Concatenated codes and Turbo codes, discussed later, have such a Shannon capacity approaching property.

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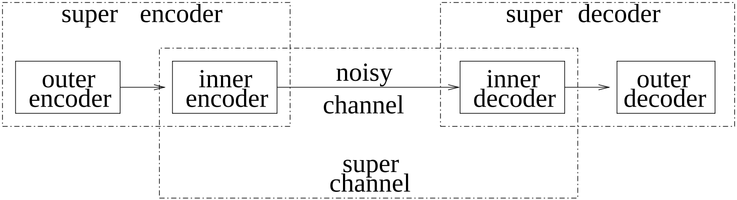

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In 1965 concatenated codes were considered as unfeasible. However, already in 1970s technology has advanced sufficiently and they became standardize by NASA for space applications.

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A code concatenated codes C_{out} and C_{in} maps a message

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CHANNELS (STREAMS) CODING

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A breakthrough came when D. Forney, in his PhD thesis in 1972, showed that so called concatenated codes could be used to achieve exponentially decreasing error probabilities at all data rates less than the Shannon channel capacity, with decoding complexity increasing only polynomially with the code length.

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Codes with lower code rate can usually correct more errors. Consequently, the communication system can:

- **operate with a lower transmit power;**
- **transmit over longer distances;**

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- **transmit at a higher data rate.**

CHANNEL CAPACITY

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By the **noisy-channel Shannon coding theorem**, the channel capacity of a given channel is the limiting code rate (in units of information per unit time) that can be achieved with arbitrary small error probability.

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The joint distribution $P_{X,Y}(x,y)$ is then defined by

$$P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x),$$

where $P_X(x)$ is the marginal distribution.

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Central problem of channel encoding: encoding is usually easy, but decoding is usually hard.

APPENDIX II.

LOCALLY DECODABLE CODES -I

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Locally decodable codes have a variety of applications in cryptography and theory of fault-tolerant computation.

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Moreover, this can be done by picking at random only three bits of the received message and combining them in a right way.

TURBO CODES

EXAMPLE from SPACE EXPLORATION

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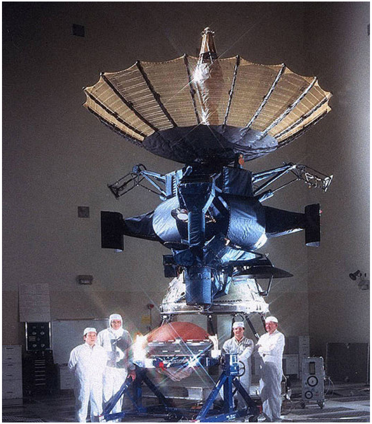

At the very beginning of the Galileo mission to explore Jupiter and its moons in 1989 it was discovered that primary antenna (deployed in the figure on the top) failed to deploy,

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Nowadays when so called iterative decoding is used concatenation of even very simple codes can yield superb performance.

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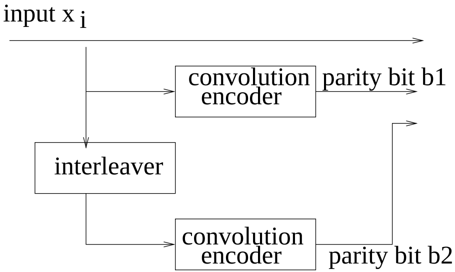
A Turbo encoder is formed from the parallel composition of two (convolution) encoders separated by an interleaver.



EXAMPLES of TURBO and CONVOLUTION ENCODERS

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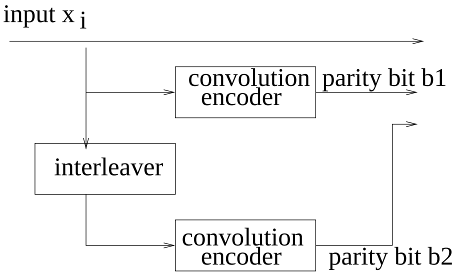

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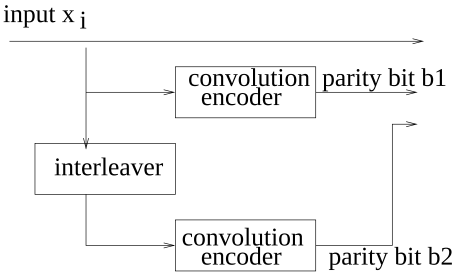
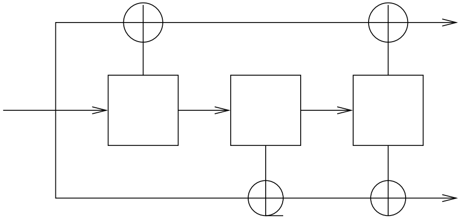


and a convolution encoder



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However, after the inverse permutation the output actually will be

$$c.n.j.200k.$$

which is quite easy to decode correctly!!!!

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Decibel

The decibel **dB** is a number that represents a logarithm of the ration of two values of a quantity (such as value $dB = 20\log(V_1/V2)$

A decibel is a relative measure. If E is the actual energy and E_{ref} is the theoretical lower bound, then the relative energy increase in decibels is

$$10\log_{10}\frac{E}{E_{ref}}$$

Since $\log_{10} 2 = 0.3$ a two-fold relative energy increase equals $3dB$.

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- A turbo code can be seen as a refinement of concatenated codes plus an iterative algorithm for decoding.