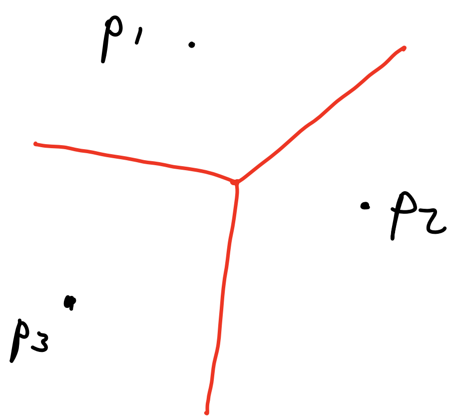
Lecture 11 - Voronoi diagrams

Post office problem:

- city with postoffices $P = \{p_1, \ldots, p_n\}$.
- Divide city into regions $V(p_i)$ around each post office p_i such that each point in $V(p_i)$ is closest to p_i.



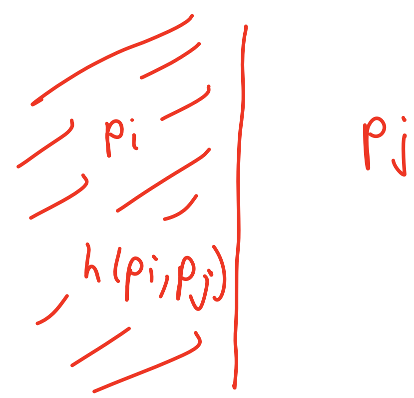
- Given $P=\{p_1,\dots,p_n\}$ the <u>Voronoi diagram</u> (V-diagram) is subdivision of plane $\mathbb{R}^2$ into n regions

$$\underbrace{V(p_i)}_{\text{Voronoi cell}} = \{q \in \mathbb{R}^2 : d(q,p_i) \le d(q,p_j) : j \ne i\}$$

- Let $h(p_i,p_j) = \{q : \underbrace{d(q,p_i) \le d(q,p_j)}_{\text{halfplane}}\}$
- Then

$$V(p_i) = \bigcap_{j \ne i} h(p_i,p_j)$$

& as an intersection of half-planes is a <u>convex polygon</u>.



- By <u>Lecture 5</u>, intersection of n half-planes takes time $O(n \log n)$
- Using this, to calc n Voronoi cells takes time $O(n^2 \log n)$

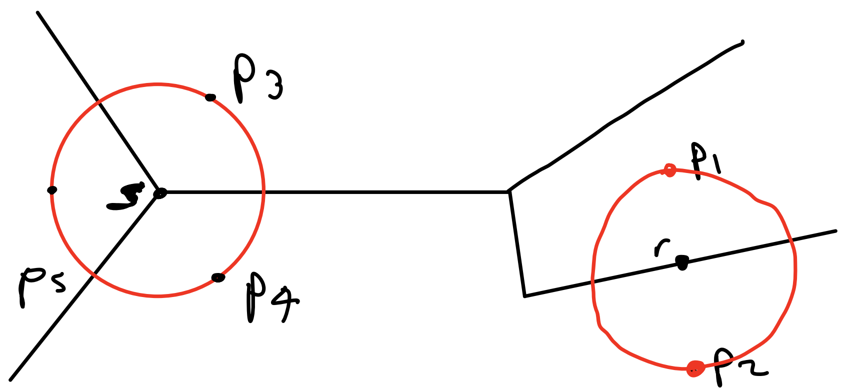
Today: <u>Faster algorithm</u> - <u>$O(n \log n)$</u> (Fortune's algorithm)

Observations

- It is not hard to see that:

① $r \in \mathbb{R}^2$ lie on an edge of V-diagram $\Longleftrightarrow$ r is equidistant from its nearest two points of P

② r is a vertex of V-diagram $\Longleftrightarrow$ r is equidistant from its nearest three points.



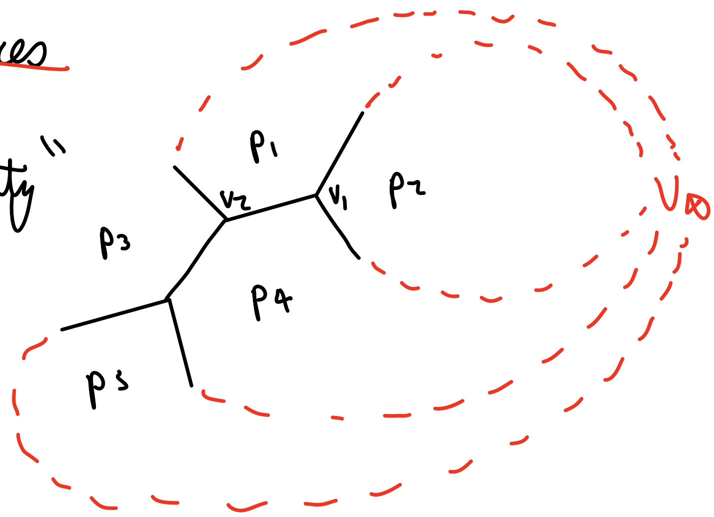
See Fig 9.3 in E-Learning.

Theorem) Any V-diagram for a set of $n \geq 3$ points (not on a line) has at most $2n-5$ vertices & at most $3n-6$ edges.

Proof) m = no of vertices

h = no of edges

- Add vertex v_∞ at "infinity" as endpoint for all half-lines.
- Obtain connected planar graph with n faces, $m+1$ vertices, h edges.



- Euler's formula $(m+1) - h + n = 2$.
- Degree of vertex ≥ 3 by ② above. Generally, sum of degrees $= 2h$ so

(*) $2h \geq 3(m+1)$.

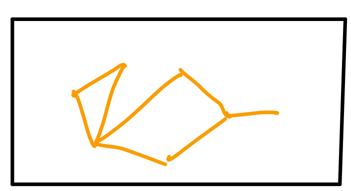
Sub $h = m+n-1$ (from Euler) into (*) gives

Aside: Euler formula

$V - E + F = 2$

vertices, edges, faces

$7 - 8 + 3 = 2$



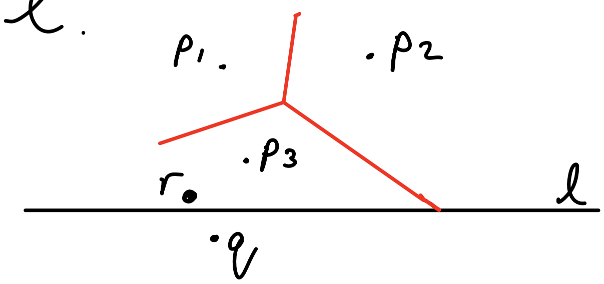
$2m + 2n - 2 \geq 3(m+1) \Rightarrow m \leq 2n - 5$.

Sim. subbing $m = h - n + 1$ into $*$ gives $h \leq 3n - 6$. $\square$

Sweep-line algorithm - Fortune's algorithm

(See animation "Voronoi tesselation" on YouTube by K. Schaal)

- Would like to use sweep-line approach to compute V-diag above sweep-line ℓ.



Problem: new point q below sweep-line can change the Voronoi cells above -
i.e. in example, $r \in V(p_3)$ before ℓ passes q, $r \in V(q)$ afterwards.

- This problem can only occur at point r for which $d(r,\ell) < d(r,p)$ for each $p \in P$ above sweepline.

- For p above ℓ,

let $\alpha^+(p,\ell) = \{ x : d(x,p) \leq d(x,\ell) \}$

$\alpha(p,\ell) = \{ x : d(x,p) = d(x,\ell) \}$

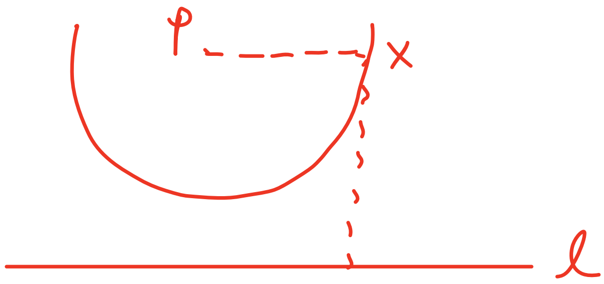
- By the above reasoning, we can correctly compute V-diagram above ℓ in the region

$$\bigcup_{p \text{ above } \ell} \alpha^+(p,\ell)$$ as any point

in this region is closer to some p above

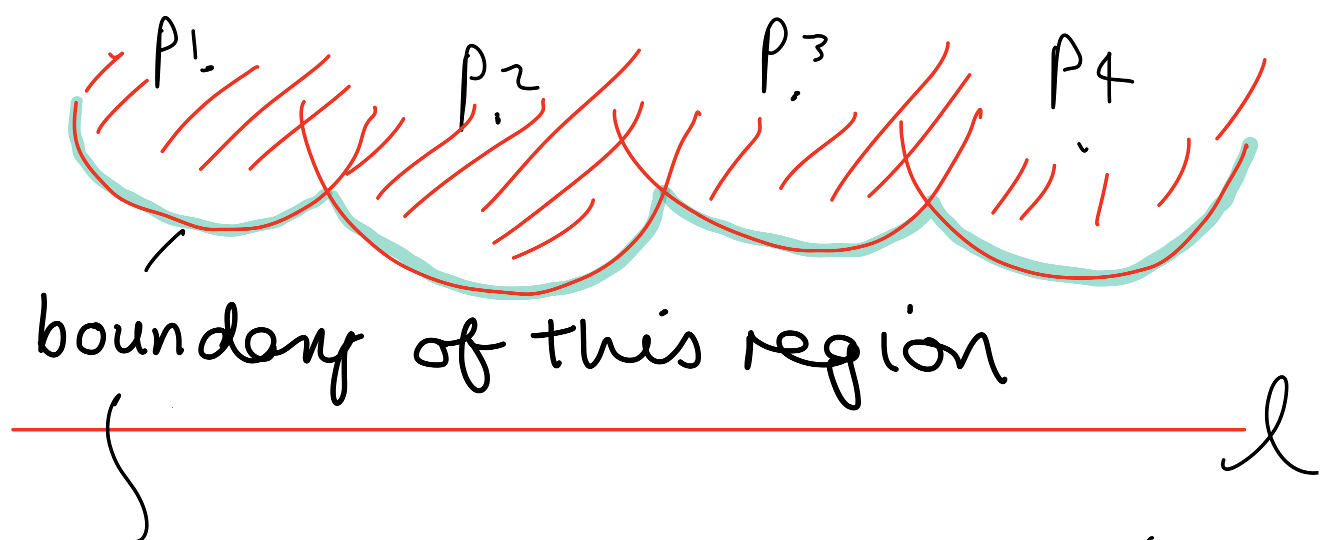
l than it is to l.

- Each $\alpha(p, l)$ is a parabola:



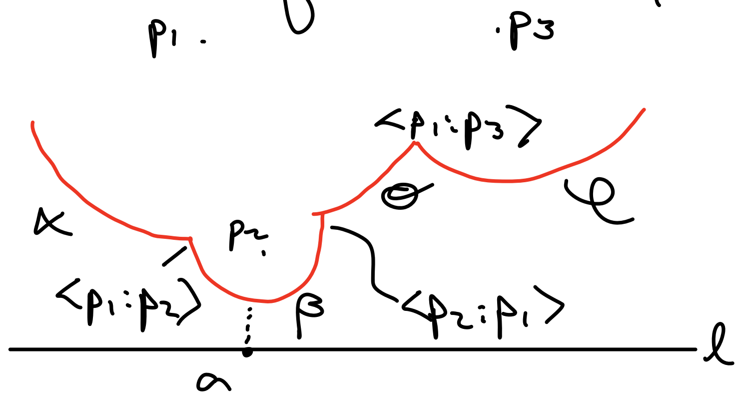
w/ focus p & directrix l

& the union $\bigcup_{p \text{ above } l} \alpha^+(p, l)$ consists of arcs of paraboloi & the region above

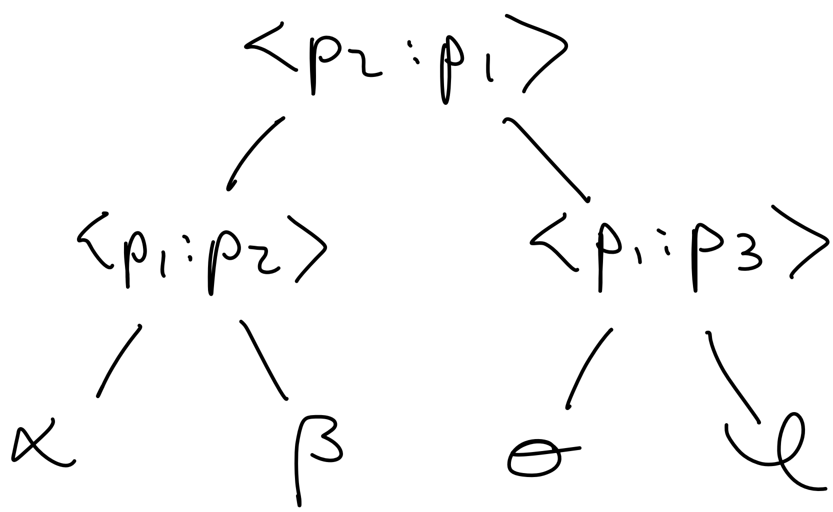


is called the <u>beach-line</u>.

At ℓ, store beach-line using a balanced binary tree:



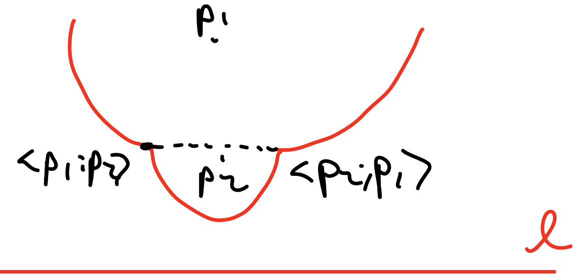
- leaves = arcs of beach line.
- Internal nodes of tree: $\langle p_i : p_j \rangle$ represent "break points" on beach-line at which paraboloi around p_i & p_j meet, with arc of p_i to left & arc of p_j to right.



- Given a point a on ℓ, can search for arc of beach-line above a.

Creating edges of Voronoi diagram

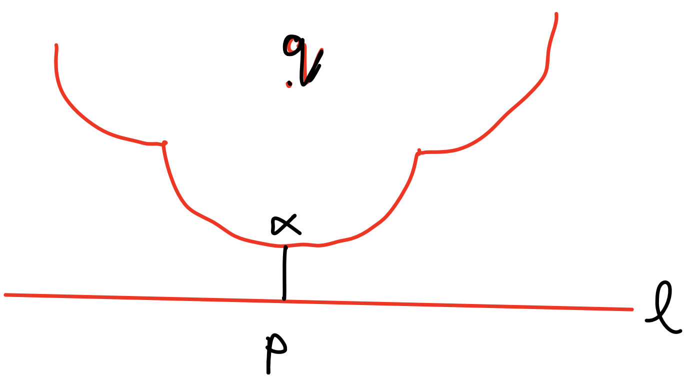
- At breakpoint $r = \langle p_1 : p_2 \rangle$, we have $d(r, p_1) = d(r, \ell) = d(r, p_2)$.

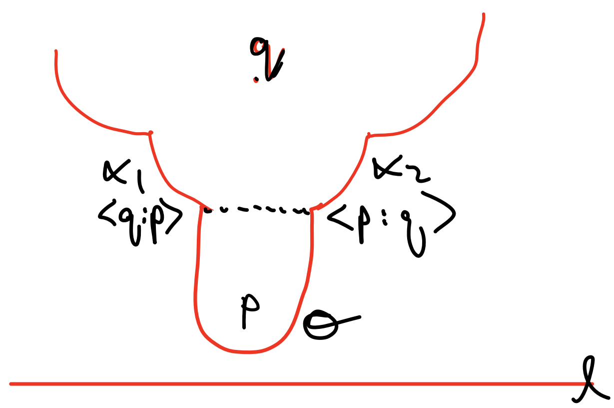


- This means that r lies on an edge of the V-diagram.
- If $\langle p_2 : p_1 \rangle$ is on beach-line, then it has same distance from p_1 & p_2.
- Therefore the edge from $\langle p_1 : p_2 \rangle$ to $\langle p_2 : p_1 \rangle$ will lie on V-diagram.
- Main technique for constructing edges on V-diagram.

Key question

- When do arcs <u>appear</u> or <u>disappear</u> from the beach line?
- A new arc appears just when the sweep-line passes a point of P.
- See Fig 9.6 & 9.7.

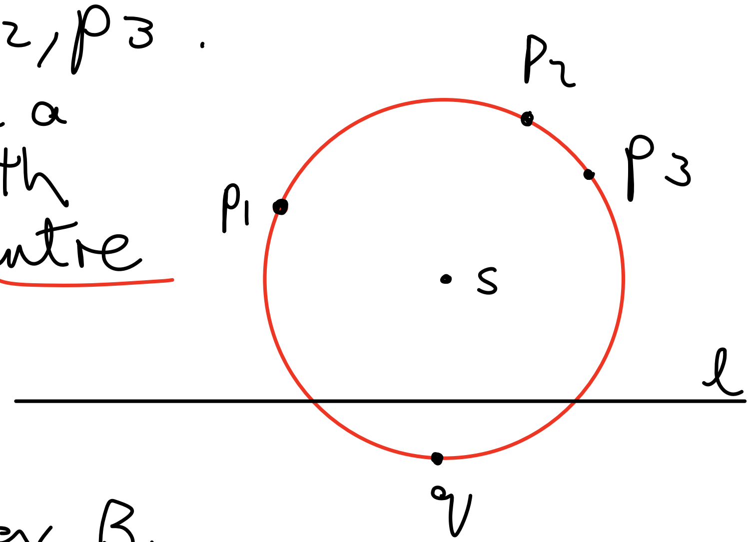




- In this case, we <u>add edge</u> between the new breakpoints.

When does an arc disappear?

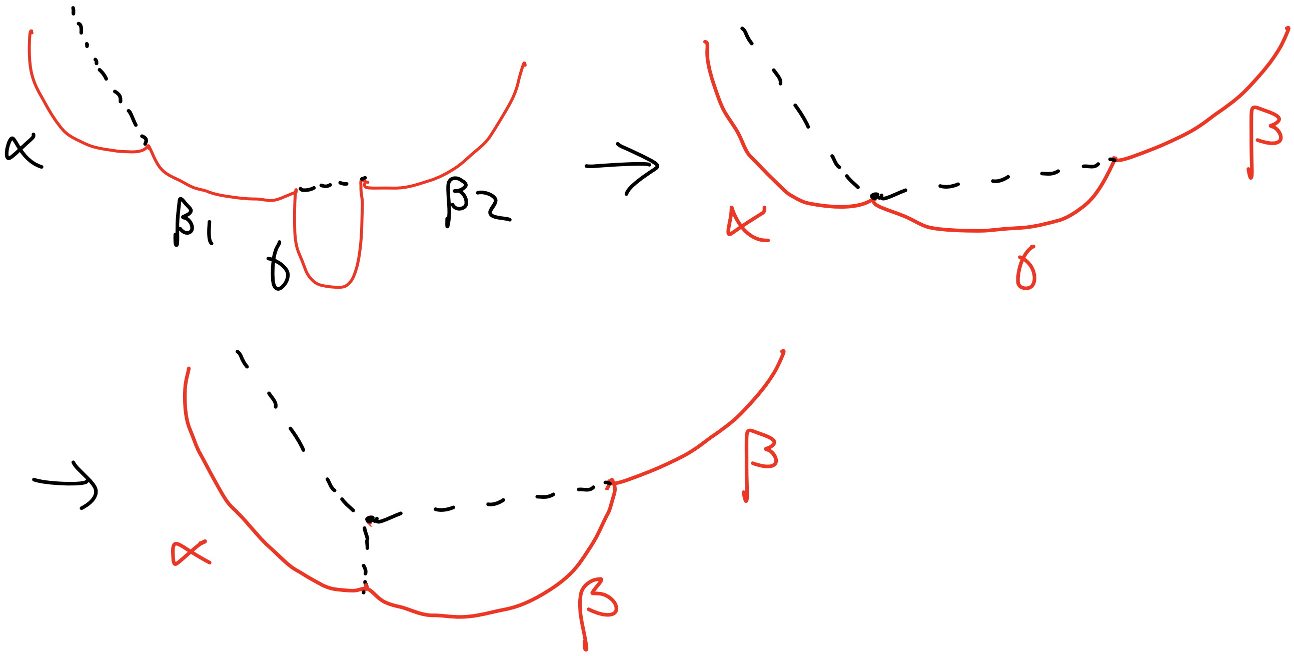
- Intuitively, when squeezed out by two adjacent arc. See Fig 9.8.
- Consider consecutive arcs α, β_1, γ with foci p_1, p_2, p_3.
- Any 3 points lie on a unique circle, with lowest point q & centre s.
- If q lies below ℓ, it is called a circle event for β_1.



- Because when ℓ passes q,

$$d(s,p_1) = d(s,p_2) = d(s,p_3) = d(s,\ell)$$

& so s lies on all 3 arcs.
- In particular, α & γ meet at s & β_1 disappears.
- A new vertex is created at s.



Algorithm: key structures

- event queue Q (actually a bin heap)
- bbtree T for beach line
- DCEL for Voronoi diagram.

• At beginning, add all points of P to Q - called site events.

• At site event p: remove p from Q
  - create new arc α of beach line,
  - new edge of V. diagram
  - search for circle events for neighbours of α & add to queue.

- At circle event q for β,
  - remove q from Q,
  - remove β from T
  - create new vertex of V-diagram
  - remove circle events for neighbours of β & search for new ones.

Complexity: $O(n \log n)$

- This lecture mainly describes the geometric ideas in constructing the Voronoi diagram.
  Further details on implem. can be seen in E-Learning & eg. the thesis linked at the end.