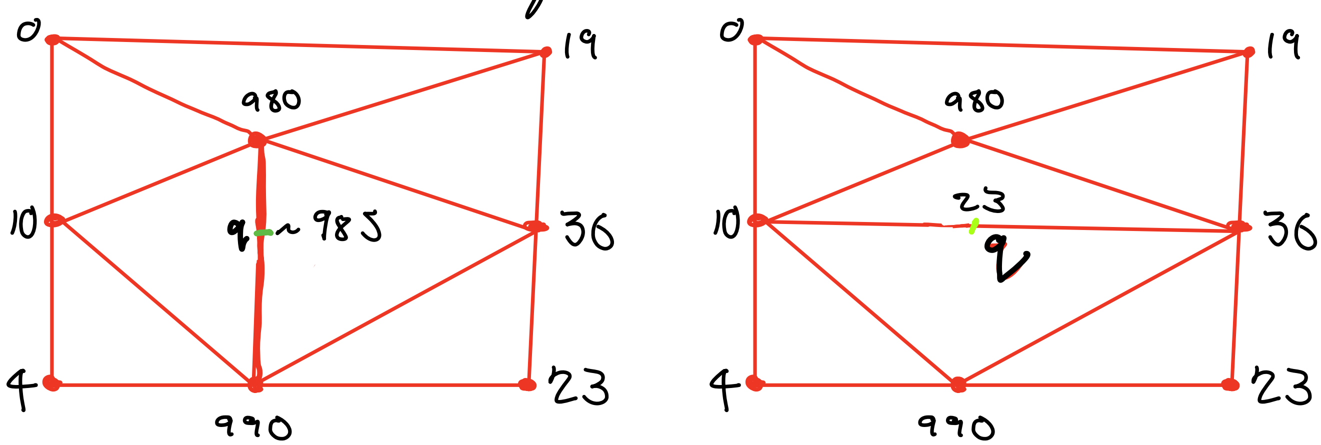
Delauney Triangulation

- Consider $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ whose values we know for finitely many points $P \subseteq \mathbb{R}^2$.
- How can we approximate f on whole plane $\mathbb{R}^2$?
- One way: find triangulation of convex hull of P, & define f linearly on each triangle.



- Picture these as representing mountaineous regions.
- First case - mountain ridge at q
  Second case - valley through q
- Which is better triangulation?
  No one answer to this:
  aesthetically, might say first case is better because q is determined by nearby points ~
  no long thin triangles ~
  avoid small angles

We will construct triangulations which avoid small angles.

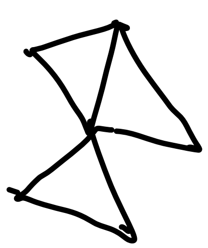
Proposition

Let P be n points in plane & suppose the convex hull of P has K edges. Then any triang. of convex hull has

$2n-2-K$ triangles &
$3n-3-K$ edges.

Proof

m = no. of triangles.
E = no of edges = no of edges app. on 1 triangle (K)



+ no. appearing on two triangles (ℓ)

Then
$$3m-\ell = E = K+\ell$$
So
$$2E = 3m-\ell+K+\ell$$
$$= 3m+K.$$

Now use Euler's formula
$v - E + F = 2$:
$n - E + (m+1) = 2$.
Subbing first formula
into second,
$m = 2n - 2 - k$ &
$E = 3n - 3 - k$. $\square$

Therefore any triangulation T of P has m triangles & so $3m$ angles:

we order these angles in a sequence

$\alpha(T) = (\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{3m})$

Define

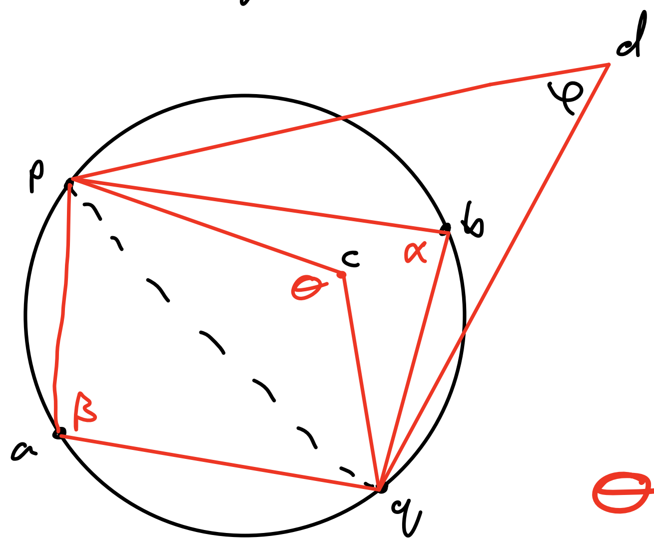
$\alpha(T) < \alpha(T')$ lexicographically

that is, $\exists i$ st $\forall j < i$ we have

$\alpha_j = \alpha'_j$ but $\alpha_i < \alpha'_i$.

- Triangulation is angle optimal if maximal wrt this ordering.
- We will not nec. find angle opt. triang. but will work with weaker notion of Legal / Delaunay triangulations.

Legal Triangulations



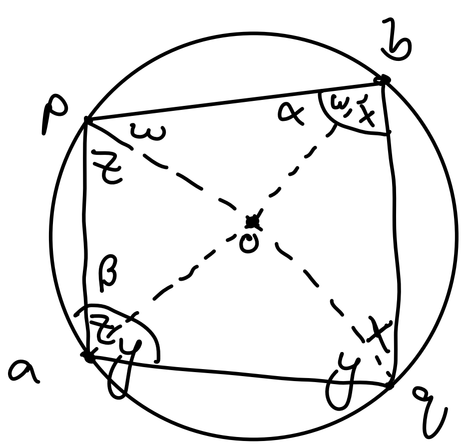
$\theta > 180 - \beta$

$\int \; 180 - \beta > \varphi$

- In circle, $\alpha + \beta = 180°$
- If c lies inside circle, $\theta > \alpha$.
- For d outside circle, $\alpha > \varphi$.

Proof that $\alpha + \beta = 180°$

- Triangles connected to centre o are isoceles, so get w, x, y, z as depicted.

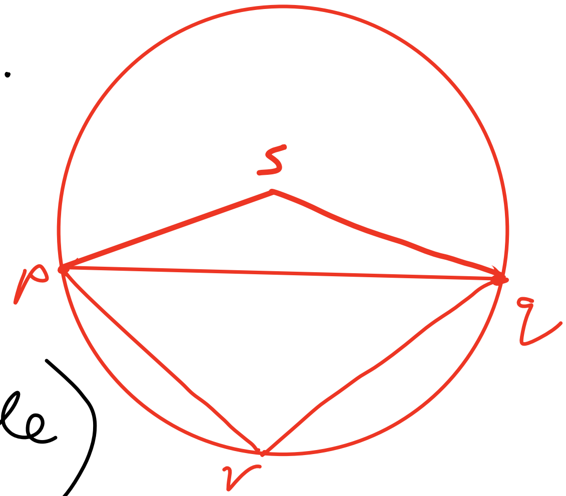


$2(w + x + y + z) = 360$

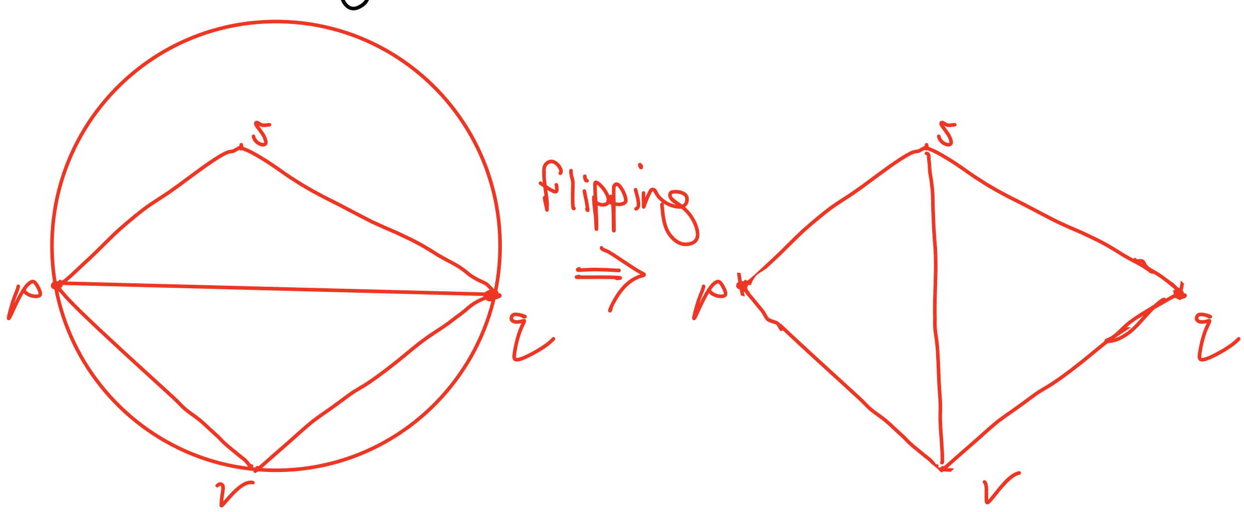
so $\underbrace{w + x + y + z}$

$\alpha + \beta = w + x + y + z = 180°$

- Consider edge $\vec{pq}$ in triangulation.
- If $\vec{pq}$ not on boundary, then there are 2 triangles pqr & pqs.
- Say pq is illegal if s lies strictly inside circle circumscribing pqr (ie. $\angle psq > 180 - \angle prq$)



- (Equiv. r lies inside circle containing pqs.)
- Otherwise pq is legal.
- A legal triangulation is one in which all edges are legal.
- Given an edge $\vec{pq}$ as above, we can "flip it" to an edge $\vec{rs}$ giving a new triangulation



lemma) Let T have illegal edge $\vec{pq}$.
Then the flipped edge $\vec{rs}$ is legal in the new triangulation T' and $\alpha(T) < \alpha(T')$.

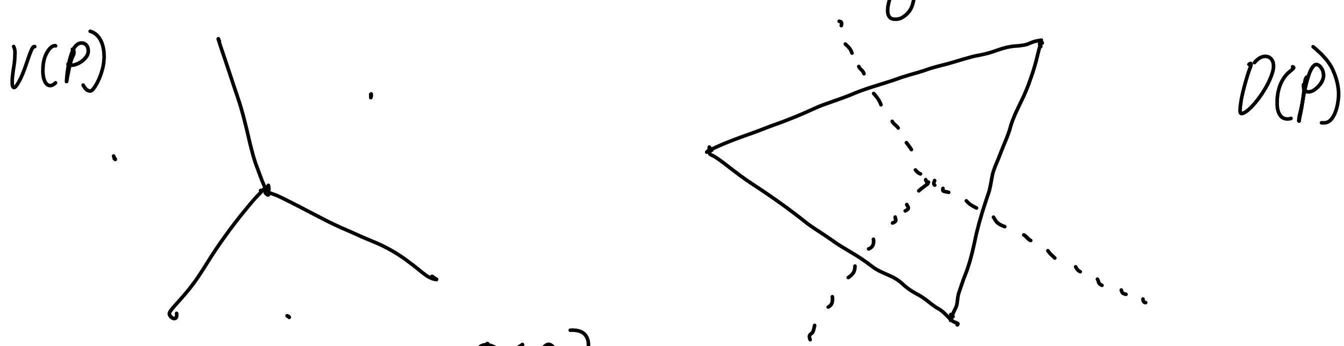
- See lemma 10.3 & proof in E-learning.

- Hence one can legalise triangulations by flipping illegal edges, & this is what our algorithm will do.

- Clearly angle optimal triangs ~~are~~ legal - else we could increase position in ordering by flipping edges.

Alternative approach - Delauney Triang.

- From P form Voronoi diag $V(P)$ (see L11)



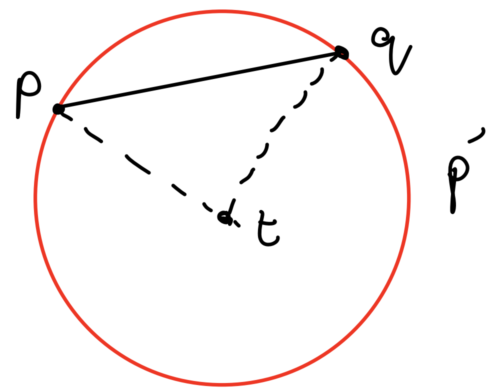
- Delauney graph $D(P)$ is dual graph to Voronoi:
  - same vertices as P (one for each face of $V(P)$)
  - an edge from p to $q \Longleftrightarrow V(p)$ & $V(q)$ share common edge.
  - Faces of $D(P)$ correspond to vertices of $V(P)$.

(See Fig 10.7 for example)

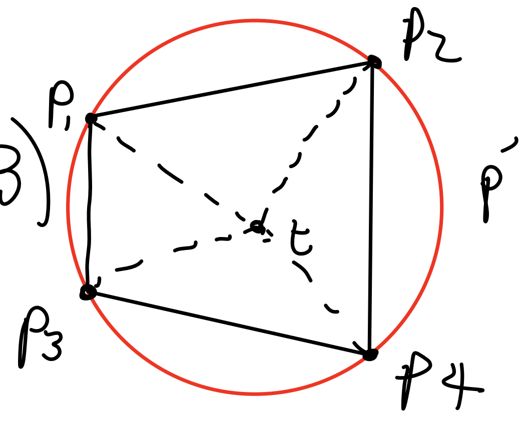
Can describe $D(P)$ in elementary terms

- From Voronoi Diag (L11),
$U(p)$ & $U(q)$ share edge $\Leftrightarrow$ $\exists t$ s.t.
$d(t,p) = d(t,q) \leq d(t,p')$ for all $p' \in P$.

- In other words, p, q lie on boundary of circle containing no other pts of P in interior.

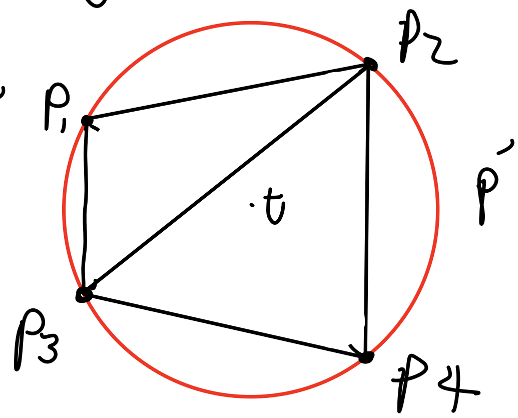


- The faces of $D(P)$ are polygons transcribed on circles with centre t, having same distance to each vertex (at least 3) & no point of P in interior.



Def) A DeLauney Triangulation is any triangulation of Delauney graph $D(P)$.

- So De Launey Triang. is obtained by triangulating these polygons.
- By its construction, it has the following property:



(*) Let $p_i p_j p_k$ be a triangle in a De Launey Triangulation. Then the circle transcribing triangle contains no points of P in its interior.

- This implies that each edge in De Launey Triangulation is Legal.

Th 10.7 from E-Learning) De Launey Triangulations $\equiv$ Legal Triangulations

Algorithm:

- Could calculate V. diagram of P, calculate its dual & triangulate it.
- We will use legalisation.

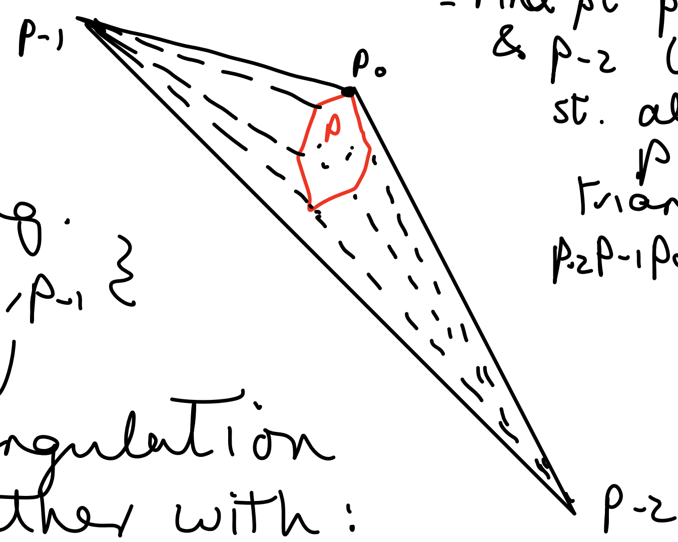
Naive version:

- Find any triangulation of convex hull of P, go through edges flipping them if illegal.
- Process must terminate, since flipping illegal edges increases pos^n of triangulation in ordering & only finitely many triangulations.

Randomised incremental algorithm

Step 1 p_0 max point from P in lex ordering (y coord first)

- Find pt p_{-1} (above left) & p_{-2} (below right) st. all points of P belong to triangle $p_{-2}p_{-1}p_0$.



Then a legal triang. of $P \cup \{p_{-2}, p_{-1}\}$ consists of legal triangulation of P together with:

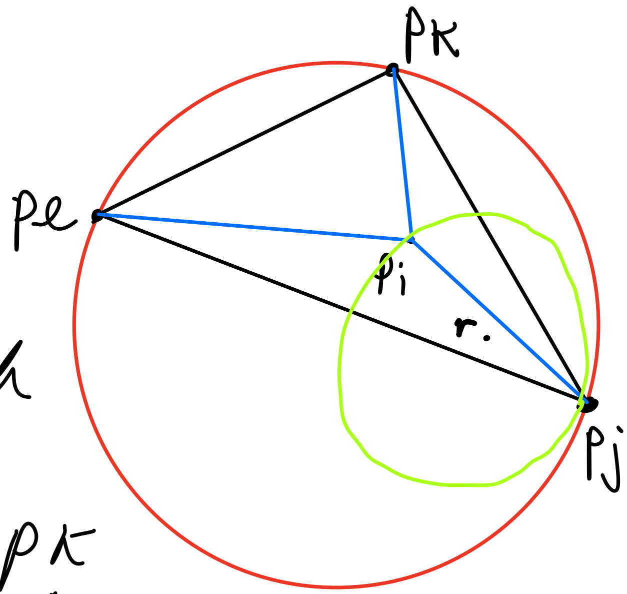
- an edge from p_{-1} to each pt of left bound,
- an edge from p_{-2} to each pt on right boundary

Step 2) Suppose we have legal triangulation T_{i-1} of

$P_{i-1} = \{p_{-2}, p_{-1}, p_0, ----, p_{i-1}\}$ for $i \geq 1$.

- Order of p_i is randomised.
- Use search structure D_{i-1} to find a Triangle or edge in T_{i-1} where p_i lies.

- Create new triangles as depicted.



- Now, each edge p_ip_e, p_ip_j, p_ip_k is legal: eg. p_ip_j.

Draw circle homothetic to larger one with chord p_ip_j. Then the centre r of circle is eq. from p_i & p_j & circle contains no pts of P_i apart from these, so p_ip_j is edge of Delauney graph $\Rightarrow$ legal.

- It may happen some edges $p_\ell p_k$ etc become illegal - we have to repeat & legalise these by flipping them.

??

Step 3

Remove p_{-2}, p_{-1} & all edges connected to them.

Search structure

Oriented graph - leaves are triangles of triangulation.

- inner nodes are triangles of prev. stages of triangulations.

(See Fig 10.17, 10.18)

Complexity: expected time $O(n \log n)$.