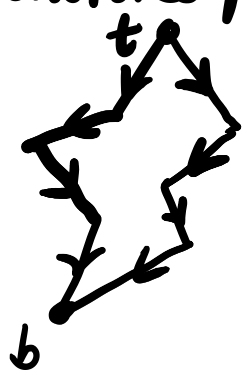
# Lecture 5

Last time,

monotone polygons:




both paths from top to bottom are decreasing wrt. lex. ordering

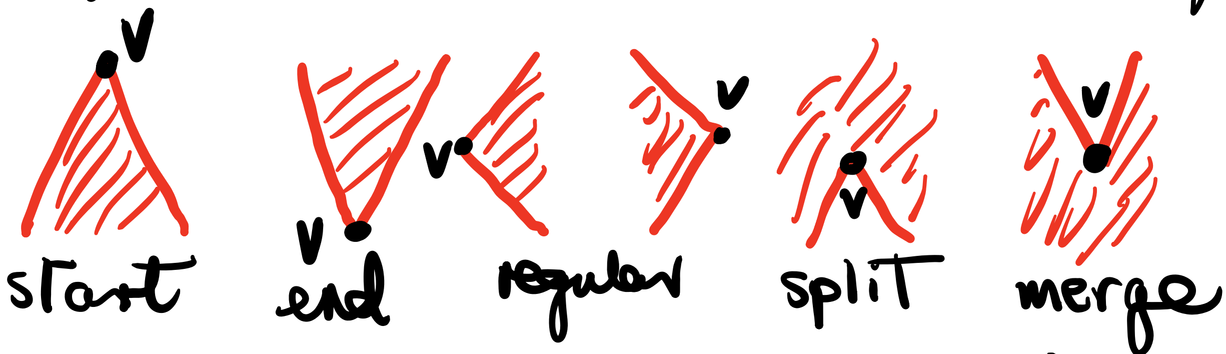
$a > b \Leftrightarrow a_y > b_y$ or $(a_y = b_y \ \& \ a_x < b_x.)$

Algorithm: Triangulate simple polygon:

1. Divide it into y-monotone parts
2. Triangulate monotone polygon.

- Last time, did 2 time $O(\log n)$.
- This week, we do 1 in time $O(n \log n)$.

# Types of vertices vs monotonicity



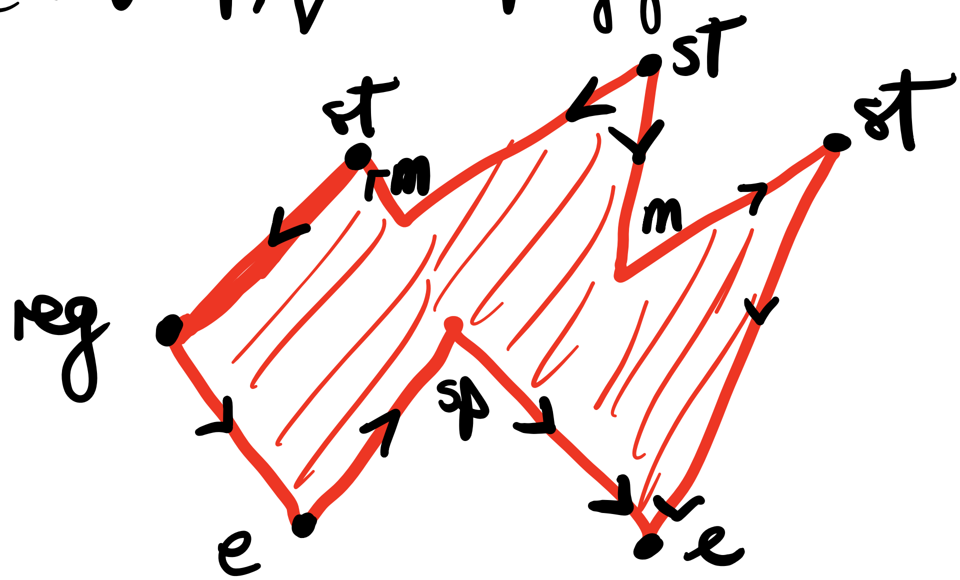

Start : $v > p, q$ (adjacent vertices) & has polygon below.

End : $v < p, q$ & has polygon above.

Reg : $p < v < q$ or $q < v < p$.

Split : $v > p, q$ & polygon above

Merge : $v < p, q$ & polygon below.



- Recall $P$ is y-monotone (monotone wrt $y$ axis) if each horizontal line intersects $P$ in 1 connected component - $\phi$, a pt or a segment.

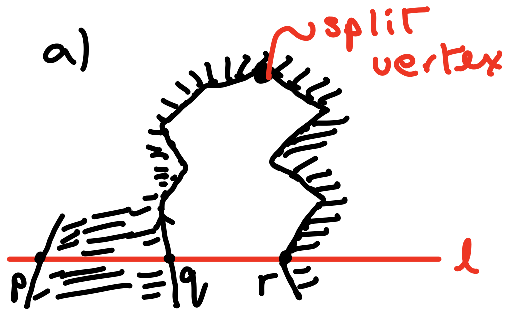
## Theorem

A simple polygon is y-monotone $\iff$ it contains no split or merge vertices.

## Proof

- If $P$ contains a split vertex 

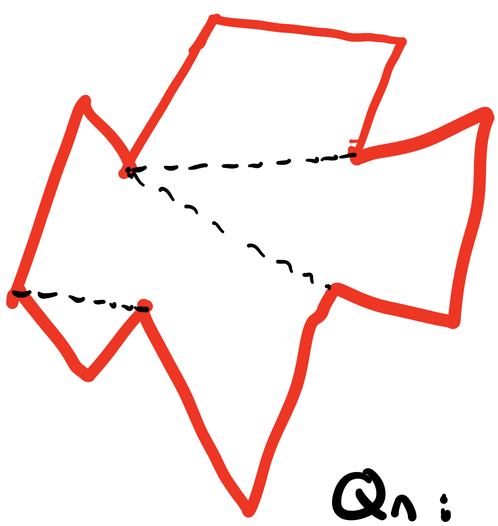
the line $\ell$ splits it into 2 components $\Rightarrow$ it is not y-monotone. The merge vertex case is similar.
- Conversely, suppose $P$ is not y-monotone, so there is horizontal line $\ell$ which intersects $P$ in more than one connected component.
- Can assume leftmost component of $P \cap \ell$ is a segment, not a point (else, move $\ell$ slightly vertically).
- Let $p$ be left pt & $q$ right point of segment
- Starting at $q$, follow boundary of $P$ so $P$ lies to left of boundary.
- Then at a point $r$, boundary of $P$ intersects $\ell$ again.
- Two cases : a) $p \neq r$ & b) $p = r$.





- In case a) highest vertex between $q, r$ is split.
- In case b), follow boundary from $q$ in opposite direction & let $r'$ be intersection point.
- The lowest vertex between $q$ & $r'$ is then a merge vertex. $\square$

Given the above, we can break simple polygons into y-monotone ones by removing split & merge vertices.

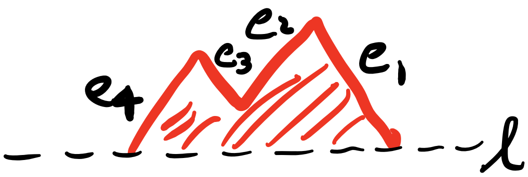


Idea: at merge vertex, draw a line downwards to a vertex
- At split vertex, draw a line upwards to a vertex.

Qn: To which vertices, do we draw lines?

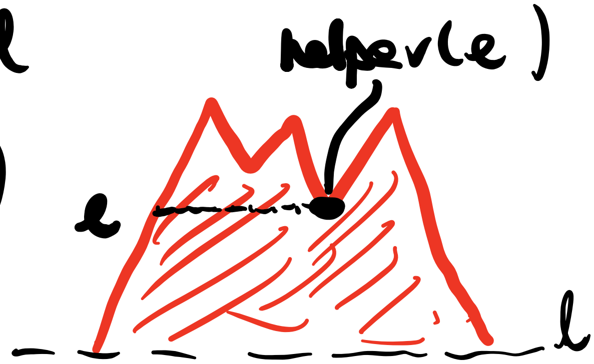
Naturally, we use a sweep-line algorithm from top to bottom.

- Polygon stored in a DCEL.
- Event queue Q (actually a bal. bin. Tree) stores vertices of polygon in lex order.
- Also bal. bin. tree T which stores edges intersecting sweep-line & having the polygon to their right.

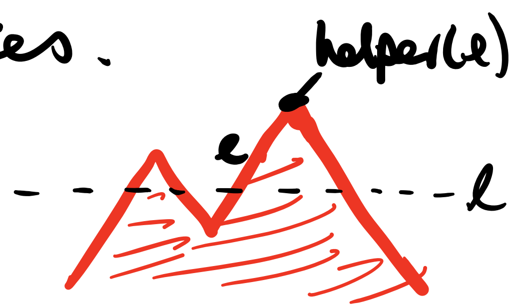



At $\ell$, $T = \{e_4, e_2\}$

- Also, with each edge $e$ in $T$ we store a vertex $p = \text{helper}(e)$:
  - helper(e) lies above $\ell$
  - horizontal segment between $e$ & helper(e) belongs to P.
  - helper(e) is lex. least vertex with these properties.




  - It may be the case that helper(e) is its upper endpoint.



# Overview of algorithm

When sweepline passes a vertex, we do some of :

- connect a vertex with helper of an edge in DCEL;
- add edges & their helpers into T;
- remove edges & their helpers from T;
- change helpers of some edges in T.

Also, we use anticlockwise enumeration of vertices & edges

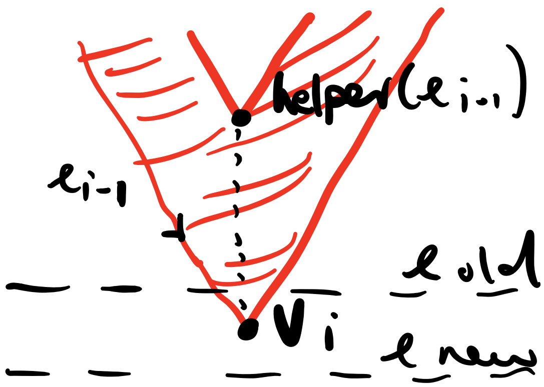

beginning from the Top (calculate using DCEL)

Cases :

**Start**   Passing start vertex $v_i$:


- Add $e_i$ to T.
- Set helper$(e_i) = v_i$

# End




- If helper($e_{i-1}$) is merge, add edge from $v_i$ to it in DCEL D.
- Remove $e_{i-1}$ from T.

# Split

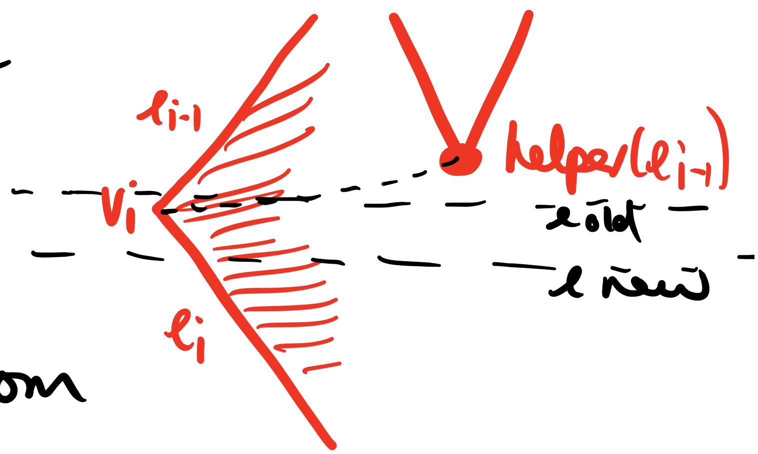

- Search T for closest edge $e_j$ to left of $v_i$.
- Add edge from $v_i$ to helper($e_j$).
- Add $e_i$ to T.
- Set helper($e_i$) = $v_i$ & helper($e_j$) = $v_i$.

# Merge

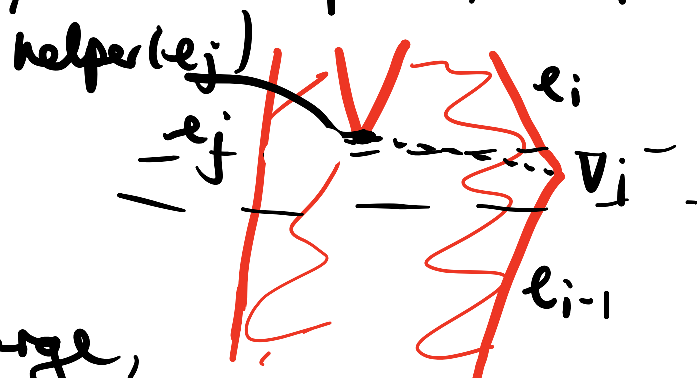

- If helper($e_{i-1}$) is merge, add edge to $v_i$ in D
- If helper($e_j$) is merge, add edge to $v_i$ in D.
- Set helper($e_j$) = $v_i$ )

## Regular vertex

- If P lies to right of $v_i$, then if helper$(e_{i-1})$ is a merge vertex, we draw a line from it to $v_i$.



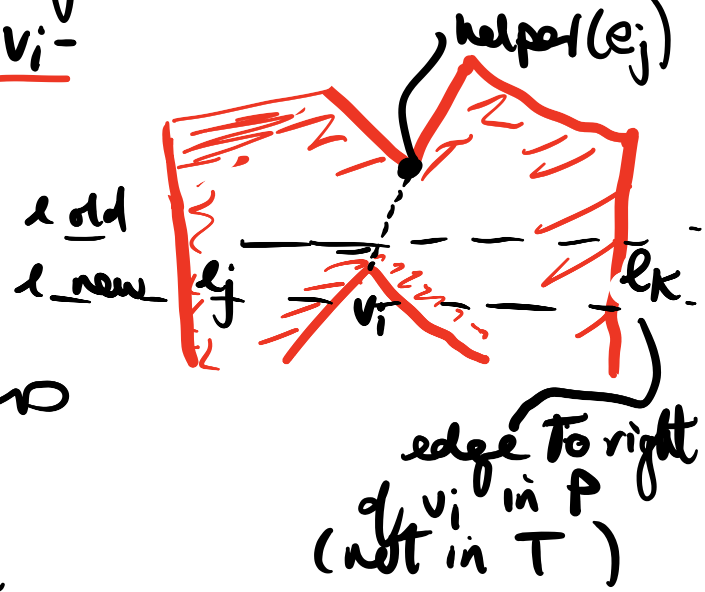

- Delete $e_{i-1}$ from T.
- Insert $e_i$ into T, with helper$(e_i) = v_i$)



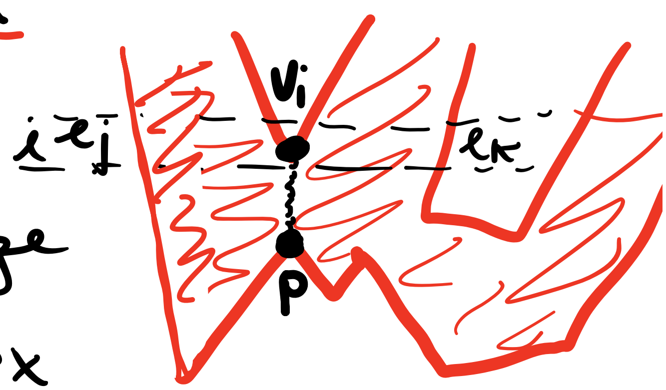

- Otherwise, P lies to left of $v_i$.
- Search T for edge $e_j$ to left of $v_i$.
- If helper$(e_j)$ is merge, draw line from it to $v_i$.
- Set helper$(e_j) = v_i$.

# Why does the algorithm work?
(Sketch)

- Why are split & merge vertices removed?
- Consider split vertex $v_i$ - is connected to helper($e_j$), the lowest vertex between its left & right neighbours $e_j$ & $e_k$




- Consider merge vertex $v_i$, left right neighbours $e_j$ & $e_k$.




- At vertex $v_i$, change helper($e_j$) to $v_i$.
- Then at max vertex p between $e_j$ & $e_k$ & below the sweep-line, we add an edge to $v_i$.

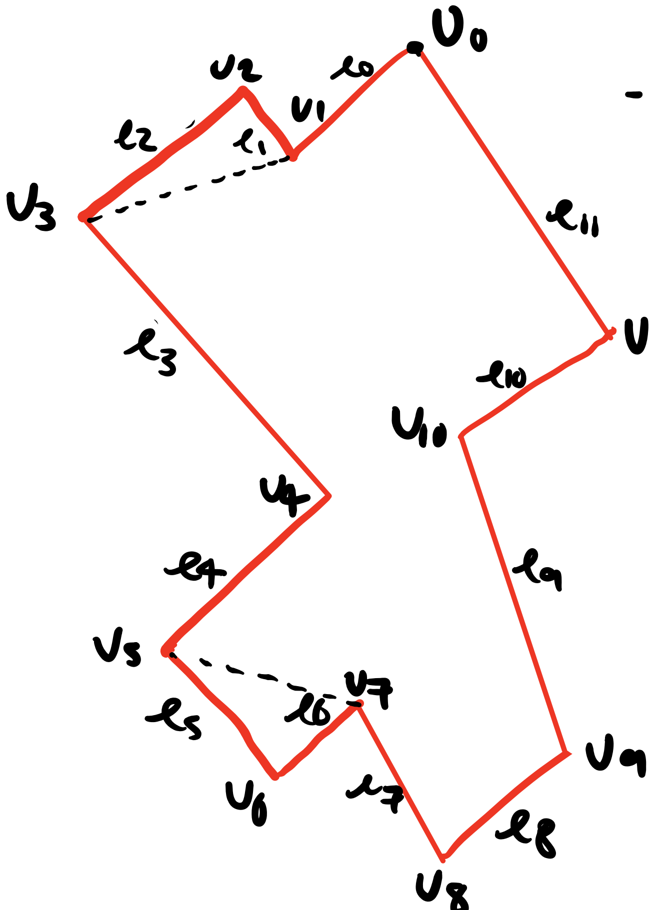
- In this way both split & merge vertices are removed.

Complexity:

- $O(n\log n)$ order vertices into Q.
- $O(n)$ to calculate anticlockwise order.
- Each event involves searching, rebalancing tree - time $O(\log n)$ - plus constant time operations: updating helpers, adding edges to DCEL.
- So time $O(n\log n)$ to handle these n events.
- Therefore complexity is

$$O(n\log n) + O(n) + O(n\log n) = O(n\log n).$$

# Example

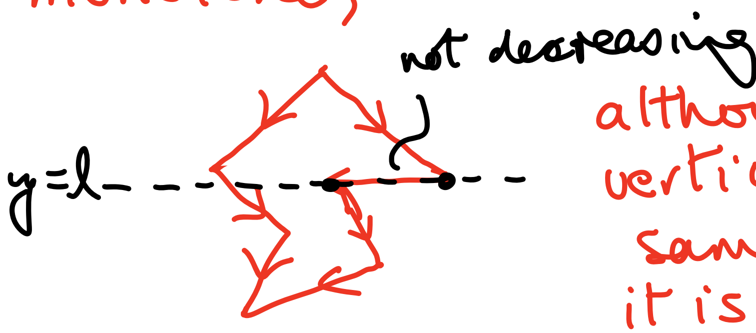



- At $v_0$, add $e_0$ to $T$ & $h(e_0) = v_0$.
- At $v_2$, also start, add $e_2$ to $T$, set $h(e_2) = v_2$.
- At $v_1$, $h(e_0) = v_0$ not merge, $h(e_2) = v_2$ not merge. Do nothing. Change $h(e_2) = v_1$.
- At $v_3$, reg. vertex, $h(e_2) = v_1$ merge. Add line $v_1$ to $v_3$. Remove $e_2$. Add $e_3$.

- At $v_{11}$, $h(e_3) = v_3$ so do nothing.
- At $v_{10}$, change $h(e_3) = v_{10}$.
- At $v_4$, remove $e_3$ & add $e_4$.
- At $v_5$, rem. $e_4$ & add $e_5$.
- At $v_7$ split, $h(e_5) = v_5$ so add line $v_5$ to $v_7$. Ch $h(e_5) = h(v_7)$. Add $e_7$.
- At $v_9$, do nothing.
- At $v_6$, remove $e_5$. At $v_8$, remove $e_7$. $\square$

Note: here we have described an alg. for dividing a simple polygon into <u>y-monotone</u> parts; last time, we described alg. for triangulating <u>monotone</u> polygon.

Not every y-monotone polygon is monotone,




although if no two vertices have same x-coordinate it is monotone.

So the algorithm for triangulating a monotone polygon will work if no two points have same x-coord.

To handle the degenerate case, one can make a small rotation to the polygon - we will not treat this here.