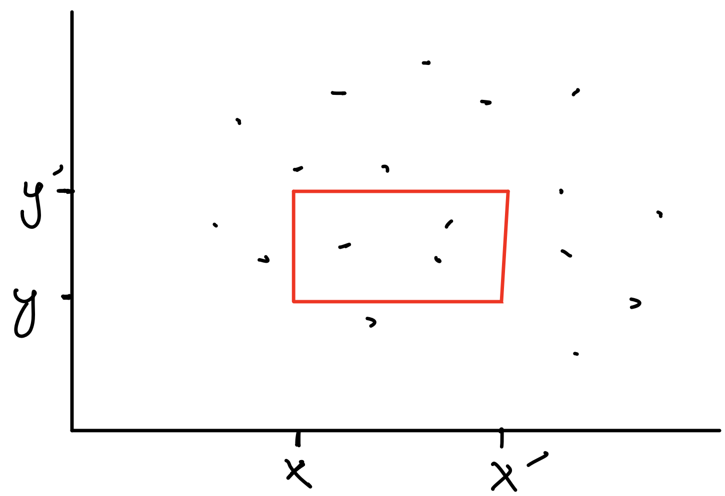
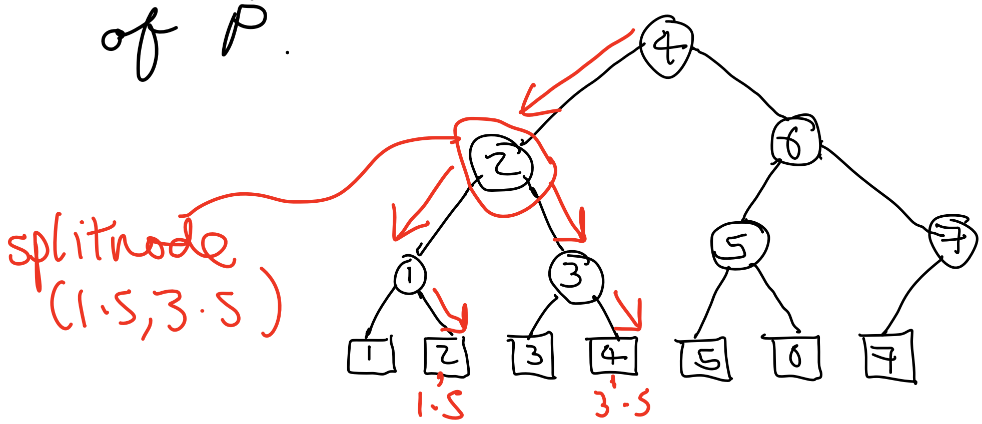
Lecture 8 - Orthogonal Range Searching

- Consider set $P \subseteq \mathbb{R}^d$ and a range $[x_1, x_1'] \times \dots \times [x_d, x_d'] \subseteq \mathbb{R}^d$.
- Find points of P belonging to the range.
- Relevant to querying databases.



1-d range searching

- $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}$ & $x \le x'$
- Points of P will be stored in a balanced binary tree whose leaves are elements of P.



- Node v stores max. value in left subtree.
- Left subtree of node v contains elements $\le x_v$, right subtree contains elts. $> x_v$.
- Given $x \in \mathbb{R}$, it determines a path from root to leaf: at node v, go left if $x \le x_v$ & right if $x > x_v$.

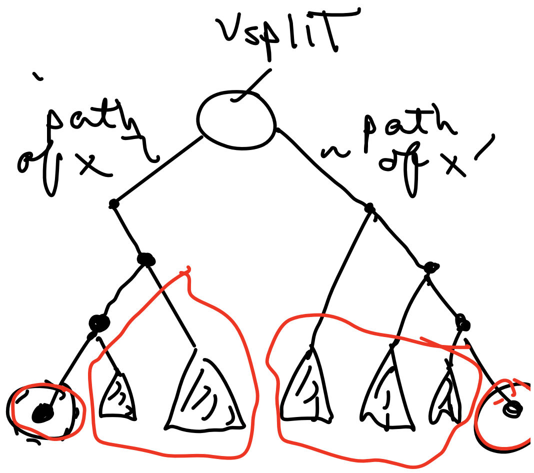
Eg: $x = 1.5$, $x' = 3.5$,

- At $x \le x'$, splitnode(x, x') is last node at which paths for x, x' agree.

Algorithm

- Input : $P = \{x_1, \dots, x_n\} \subseteq \mathbb{R}$ stored in a bal. bin. tree & $x \leq x'$
- Find points of P in range $x \leq x'$.

- Firstly, find splitnode V_{split}.
- If V_{split} is a leaf, check if it belongs to $[x, x']$ & report it, if so.
- Follow path of x from V_{split} to a leaf.
- If at node v, x moves left, we report all points in right subtree of v as solutions.
- At leaf, check if it belongs to $[x, x']$
- Similarly, follow path of x' to leaf, and when x' moves right, report left subtree. At leaf, check.



Why does this find all solutions?

- Suppose $x \leq p \leq V_{spl}$.
- Then p is reported when paths for x and p diverge, or at leaf p itself.
- Similarly if $V_{spl} \leq p \leq x'$.

- See E-Learning for pseudocode.

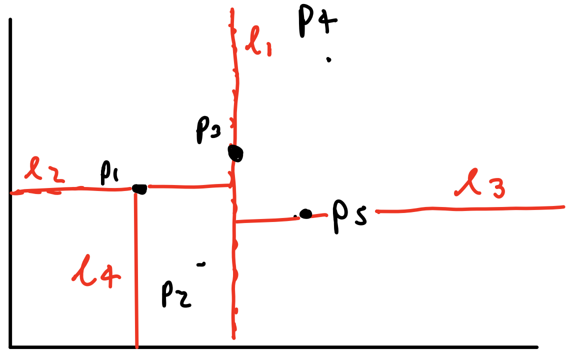
Complexity

- Time $O(\log_2 n)$ to follow path of x.
- Likewise for x'.
- Time to report k solutions is $O(k)$.

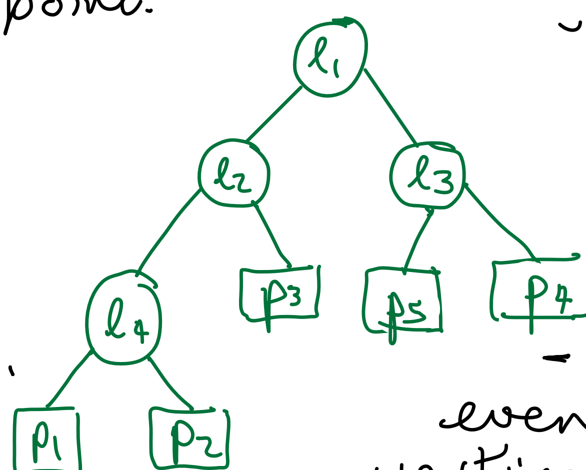
Total complexity $O(\underbrace{\log n}_{\text{no of leaves}} + \underbrace{k}_{\text{no of solutions}})$.

2-d range searching

- Set $P \subseteq \mathbb{R}^2$ (assume no 2 pts have same x or y coordinate)
- Using <u>vertical</u> line, split through <u>median</u> point ordered by <u>x-coordinate</u>. Count point on the line in <u>left region</u> (should be same number of points in either region or one more in the left)

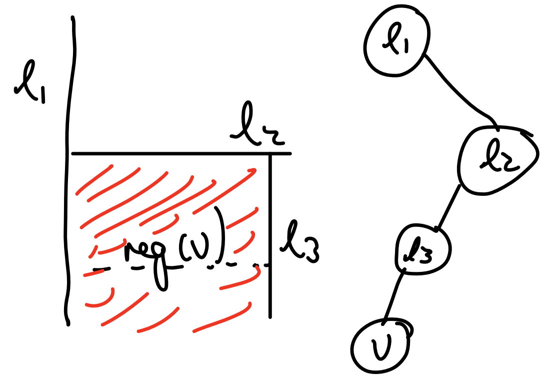


- Now split left & right regions using horizontal lines, through point with median y-coord, so lower (left) regions contains line & has same no. or 1 more point as upper (right) region.
- Repeat until each region contains 1 point.



- This data structure is called a <u>KD-Tree</u>.
- Binary tree.
- Leaves are points of P.
- Nodes of even depth store vertical lines by x-coord.
- Nodes of odd depth store horizontal lines, by y-coord.
- Takes O(n) storage, where n is no. of leaves.
- O(n log n) to const. KD-tree on n points.

- Region of a node v:
- rectangular region bordered by ancestors of v.
(ie. if v represents a line, region(v) is area which this line split into two.)



- Region (root) = $\mathbb{R}^2$
- Region (lc(v)) = Region(v) ∩ left(v)

left child

- Region (rc(v)) = Region(v) ∩ right(v)

right child

- A point p of P belongs to region(v) ⟺ p belongs to subtree under v.
- Idea: search through subtree under node v ⟺ region(v) intersects search rectangle (range).

Outline of algorithm

- Given range R & KD-Tree of points P.
- Find points of P in R

- Move downwards through tree.
- At a node v:
  - if v is a leaf, check if it belongs to P.
- otherwise, we look at lc(v), rc(v).
  - If reg(lc(v)) ⊆ R, report subtree of lc(v).
  - Else, if reg(lc(v)) intersects R, continue search of subtree of lc(v).
  - Sim, if reg(rc(v)) ⊆ R, ...

- See E-Learning for pseudocode.

Complexity $O(\underbrace{\sqrt{n}}_{n\text{ no. of points in } P} + \underbrace{k}_{\text{no. of solutions}})$

- Removing assumption that no points in P have same x or y-coordinate

Observation: did not need points to be real numbers - only needed them to be elements of a totally ordered set: so we can compare elements & find medians.

- Pass from $\mathbb{R}$ to $C = (\mathbb{R} \cup \{-\infty, \infty\})^2$ elements of form $(a|b)$
- C has lexicographic order:

$$(a|b) < (c|d) \iff a < c \text{ or } (a = c \ \& \ b < d)$$

$$\mathbb{R}^2 \longrightarrow C^2$$

$$(p_x, p_y) = p \longmapsto \hat{p} = ((p_x|p_y), (p_y|p_x))$$

- Set $\hat{P} = \{\hat{p} : p \in P\}$.
- No two points in $\hat{P}$ have same first or second coord.

- Let $R = [x, x'] \times [y, y']$,

$$\hat{R} = [(x|-\infty), (x'; \infty)] \times [(y|-\infty), (y'; \infty)]$$

- Then $p \in R \iff \hat{p} \in \hat{R}$ so only need to run our original alg (gen. to a totally ord. set) on $(\hat{P}, \hat{R})$ instead.

i.e. $(p_x|p_y) \in [(x|-\infty), (x'; \infty)]$

$\iff (x|-\infty) < (p_x|p_y) < (x'|\infty)$.

First ineq. $x < p_x$ or $x = p_x$ ~ $x \le p_x$

Second ineq. $p_x \le x'$.

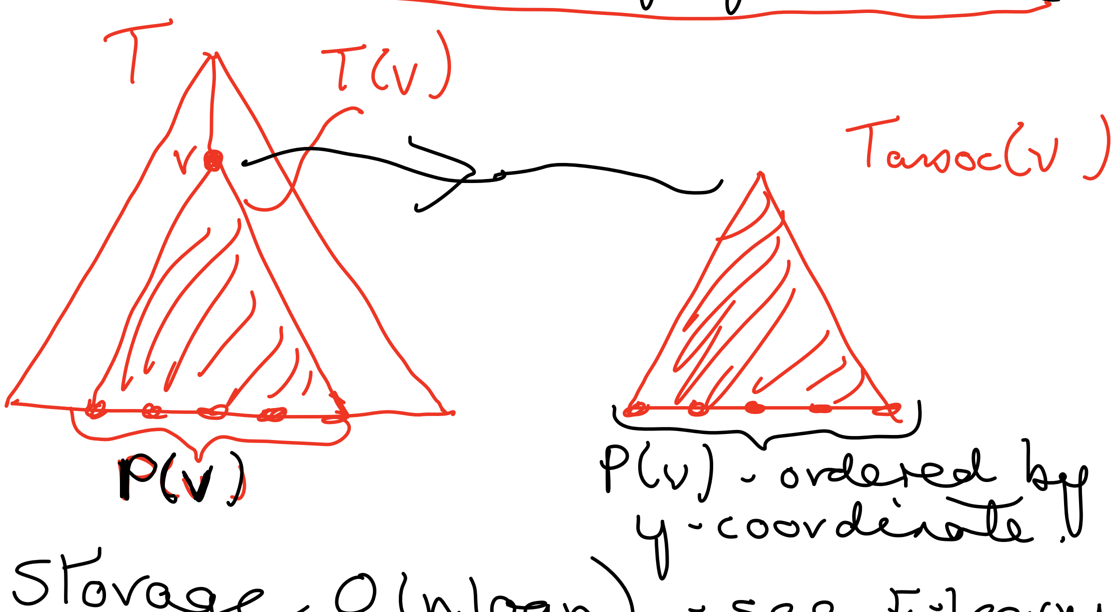
Second approach - range trees

Idea: Given $P \subseteq \mathbb{R}^2$ & $R = [x, x'] \times [y, y']$

1. Use a 1-d search to find points of P whose x-coord belongs to $[x, x']$.
2. Search amongst these points to find those whose y-coord belongs to $[y, y']$.

Data structure: range tree.

- A binary tree whose leaves are elements of P, ordered by x-coord (assume no 2 pts have same x or y coord.)
- Each node v determines subtree $T(v)$ with set of leaves $P(v)$; For each such node we have another bin. tree $T_{assoc}(v)$ with leaves $P(v)$ ordered by y-coordinate.



- Storage $O(n \log n)$ - see E-learning

Searching a range tree $\breve{T}$

$R = [x, x'] \times [y, y']$

- Look at tree ordered by x-coord, Find split node of x & x'

- If path for x moves left at v, each leaf in right subtree belongs to $[x, x']$
- Then we use a 1-d range search on $T_{assoc}(rc(v))$ to find those whose y-coord belongs to $[y, y']$.
- If v is a leaf, test whether it belongs to R
- Similarly search path of x' below split node.
- Furthermore, complexity $O(\underbrace{\log n}_{\text{no. of points}}{}^2 + \underbrace{k}_{k \text{ no. of solutions}})$
- Finally, both Kd-trees & range trees can be gen. to higher dimensions. See E-Learning for this and comparison of

two approaches