Application of the support enumeration algorithm

Consider the following game of two players (row and column player):

$$\begin{pmatrix} 5,4 & 1,3 \\ 3,2 & 3,3 \end{pmatrix}$$

- $S_1 = S_2 = \{1,2\}$
- Variables: $\sigma_1(1), \sigma_1(2), \sigma_2(1), \sigma_2(2), w_1, w_2$

Now enumerate all possible pairs of supports and solve for each of them separately. We shall use strict inequalities in the constraint 4 (see slide 96 of the lecture) to ensure that the strategies have exactly the prescribed supports. (The version from the slide allows smaller supports but is still correct and complete.)

The constraint system from the slide 96 instantiates into the following form: The constraints 1 and 2 are

$$\begin{aligned}
&\sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 \quad \{\leq,=\} \quad w_1 && (k=1)\\
&\sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 \quad \{\leq,=\} \quad w_1 && (k=2)\\
&\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 \quad \{\leq,=\} \quad w_2 && (\ell=1)\\
&\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 \quad \{\leq,=\} \quad w_2 && (\ell=2)
\end{aligned}$$

Here the concrete relations ($\leq$ or $=$) depend on the support (see below for the individual cases). The constraints in 3 do not change with supports:

$$\sigma_1(1) + \sigma_1(2) = 1 \qquad \sigma_2(1) + \sigma_2(2) = 1$$

The constraints 4 and 5 depend on the supports and will be presented individually.

Now let us enumerate the support sets.

1. $supp_1 = \{1\}, supp_2 = \{1\}$: The constraints:

$$\begin{aligned}
&\sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 = w_1 && (k=1)\\
&\sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 \leq w_1 && (k=2)\\
&\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 = w_2 && (\ell=1)\\
&\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 \leq w_2 && (\ell=2)
\end{aligned}$$

   and

$$\sigma_1(1) > 0 \qquad \sigma_1(2) = 0 \qquad \sigma_2(1) > 0 \qquad \sigma_2(2) = 0$$

   Thus, by the constraint 3, $\sigma_1(1) = \sigma_2(1) = 1$ and the above constraints reduce to

$$\begin{aligned}
5 &= w_1\\
3 &\leq w_1\\
4 &= w_2\\
3 &\leq w_2
\end{aligned}$$

   Giving $w_1 = 5$ and $w_2 = 4$ and $\sigma_1(1) = 1$ and $\sigma_2(1) = 1$.

2. $supp_1 = \{1\}, supp_2 = \{2\}$: The constraints:

$$\begin{aligned}
\sigma_2(1)\cdot 5+\sigma_2(2)\cdot 1 &= w_1 && (k=1)\\
\sigma_2(1)\cdot 3+\sigma_2(2)\cdot 3 &\leq w_1 && (k=2)\\
\sigma_1(1)\cdot 4+\sigma_1(2)\cdot 2 &\leq w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3+\sigma_1(2)\cdot 3 &= w_2 && (\ell=2)
\end{aligned}$$

and

$$\sigma_1(1)>0 \qquad \sigma_1(2)=0 \qquad \sigma_2(1)=0 \qquad \sigma_2(2)>0$$

Thus $\sigma_1(1)=\sigma_2(2)=1$ and the constraints reduce to

$$\begin{aligned}
1 &= w_1\\
3 &\leq w_1\\
4 &\leq w_2\\
3 &= w_2
\end{aligned}$$

which apparently does not have any solution. No Nash equilibrium exists with this support.

3. $supp_1 = \{2\}, supp_2 = \{1\}$: The constraints:

$$\begin{aligned}
\sigma_2(1)\cdot 5+\sigma_2(2)\cdot 1 &\leq w_1 && (k=1)\\
\sigma_2(1)\cdot 3+\sigma_2(2)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4+\sigma_1(2)\cdot 2 &= w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3+\sigma_1(2)\cdot 3 &\leq w_2 && (\ell=2)
\end{aligned}$$

and

$$\sigma_1(1)=0 \qquad \sigma_1(2)>0 \qquad \sigma_2(1)>0 \qquad \sigma_2(2)=0$$

Reduce to

$$\begin{aligned}
5 &\leq w_1\\
3 &= w_1\\
2 &= w_2\\
3 &\leq w_2
\end{aligned}$$

Which does not have a solution.

4. $supp_1 = \{2\}, supp_2 = \{2\}$: The constraints:

$$\begin{aligned}
\sigma_2(1)\cdot 5+\sigma_2(2)\cdot 1 &\leq w_1 && (k=1)\\
\sigma_2(1)\cdot 3+\sigma_2(2)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4+\sigma_1(2)\cdot 2 &\leq w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3+\sigma_1(2)\cdot 3 &= w_2 && (\ell=2)
\end{aligned}$$

and

$$\sigma_1(1) = 0 \qquad \sigma_1(2) > 0 \qquad \sigma_2(1) = 0 \qquad \sigma_2(2) > 0$$

Reduce to

$$\begin{aligned} 1 &\leq w_1 \\ 3 &= w_1 \\ 2 &\leq w_2 \\ 3 &= w_2 \end{aligned}$$

Giving $w_1 = w_2 = 3$ and $\sigma_1(2) = 1$ and $\sigma_2(2) = 1$.

5. $supp_1 = \{1\}, supp_2 = \{1,2\}$ The constraints:

$$\begin{aligned} \sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &= w_1 && (k=1) \\ \sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &\leq w_1 && (k=2) \\ \sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1) \\ \sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &= w_2 && (\ell=2) \end{aligned}$$

and

$$\sigma_1(1) > 0 \qquad \sigma_1(2) = 0 \qquad \sigma_2(1) > 0 \qquad \sigma_2(2) > 0$$

Reduce to

$$\begin{aligned} \sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &= w_1 && (k=1) \\ \sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &\leq w_1 && (k=2) \\ \sigma_1(1)\cdot 4 &= w_2 && (\ell=1) \\ \sigma_1(1)\cdot 3 &= w_2 && (\ell=2) \end{aligned}$$

which does not have a solution.

6. $supp_1 = \{2\}, supp_2 = \{1,2\}$ The constraints:

$$\begin{aligned} \sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &\leq w_1 && (k=1) \\ \sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &= w_1 && (k=2) \\ \sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1) \\ \sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &= w_2 && (\ell=2) \end{aligned}$$

and

$$\sigma_1(1) = 0 \qquad \sigma_1(2) > 0 \qquad \sigma_2(1) > 0 \qquad \sigma_2(2) > 0$$

Reduce to

$$\begin{aligned} \sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &\leq w_1 && (k=1) \\ \sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &= w_1 && (k=2) \\ \sigma_1(2)\cdot 2 &= w_2 && (\ell=1) \\ \sigma_1(2)\cdot 3 &= w_2 && (\ell=2) \end{aligned}$$

Which does not have a solution.

7. $supp_1 = \{1,2\}, supp_2 = \{1\}$ The constraints:

$$\begin{aligned}
\sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &= w_1 && (k=1)\\
\sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &\le w_2 && (\ell=2)
\end{aligned}$$

and

$$\sigma_1(1) > 0 \qquad \sigma_1(2) > 0 \qquad \sigma_2(1) > 0 \qquad \sigma_2(2) = 0$$

Reduce to

$$\begin{aligned}
\sigma_2(1)\cdot 5 &= w_1 && (k=1)\\
\sigma_2(1)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &\le w_2 && (\ell=2)
\end{aligned}$$

Which does not have a solution.

8. $supp_1 = \{1,2\}, supp_2 = \{2\}$ The constraints:

$$\begin{aligned}
\sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &= w_1 && (k=1)\\
\sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &\le w_2 && (\ell=2)
\end{aligned}$$

and

$$\sigma_1(1) > 0 \qquad \sigma_1(2) > 0 \qquad \sigma_2(1) = 0 \qquad \sigma_2(2) > 0$$

Reduce to

$$\begin{aligned}
\sigma_2(2)\cdot 1 &= w_1 && (k=1)\\
\sigma_2(2)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &\le w_2 && (\ell=2)
\end{aligned}$$

Which does not have a solution.

9. $supp_1 = \{1,2\}, supp_2 = \{1,2\}$: The constraints:

$$\begin{aligned}
\sigma_2(1)\cdot 5 + \sigma_2(2)\cdot 1 &= w_1 && (k=1)\\
\sigma_2(1)\cdot 3 + \sigma_2(2)\cdot 3 &= w_1 && (k=2)\\
\sigma_1(1)\cdot 4 + \sigma_1(2)\cdot 2 &= w_2 && (\ell=1)\\
\sigma_1(1)\cdot 3 + \sigma_1(2)\cdot 3 &= w_2 && (\ell=2)
\end{aligned}$$

Reduce (using the contraint 3) to

$$\sigma_2(1) \cdot 5 + (1 - \sigma_2(1)) \cdot 1 = \sigma_2(1) \cdot 3 + (1 - \sigma_2(1)) \cdot 3$$
$$\sigma_1(1) \cdot 4 + (1 - \sigma_1(1)) \cdot 2 = \sigma_1(1) \cdot 3 + (1 - \sigma_1(1)) \cdot 3$$

Which is solved by $\sigma_2(1) = 1/2$ and $\sigma_1(1) = 1/2$.