Homework Sheet 1

Exercise 1 (5 points) Let A, B, C be propositional variables. Determine (with proof) whether there exists a formula φ such that the formula $A \to \varphi$ is equivalent to

(a) $\varphi \to A$;

(b) $\varphi \to B$;

(c) $B \to \varphi$.

In a second step, determine whether we can choose the formula φ such that it depends on the variable C (i.e., there exist two variable assignments v, v' that agree on all variables different from C and such that $v(\varphi) \neq v'(\varphi)$).

Solution

(a) We can obviously choose $\varphi := A$. In fact, this is the only possibility (up to logical equivalence). If v is a variable assignment with $v(A) = 0$, we have

$$v(A \to \varphi) = 1 \quad \text{and} \quad v(\varphi \to A) = 1 - v(\varphi)\,.$$

Hence, if $A \to \varphi \approx \varphi \to A$, we must have $v(\varphi) = 0$.

Similarly, if v is a variable assignment with $v(A) = 1$, we have

$$v(\varphi \to A) = 1 \quad \text{and} \quad v(A \to \varphi) = v(\varphi)\,.$$

Hence, $A \to \varphi \approx \varphi \to A$ implies $v(\varphi) = 1$.

It follows that $\varphi \approx A$. In particular, it cannot depend on C.

(b) Such a formula does not exist. Let v be the variable assignment with $v(A) = 1$ and $v(X) = 0$ for all other variables. Then

$$v(A \to \varphi) = v(\varphi) \neq 1 - v(\varphi) = v(\varphi \to B)\,.$$

A contradiction.

(c) Any tautology works. Another solution is the formula $\varphi := A \vee B \vee C$, which does depend on C. To show that this formula is indeed a solution, we can use a truth table, or we can argue as follows. The formula $A \to (A \vee B \vee C)$ is true if $v(A) = 0$ and if $v(A) = 1$. Similarly, $B \to (A \vee B \vee C)$ is true if $v(B) = 0$ and if $v(B) = 1$. Hence, both formulae are tautologies.

Exercise 2 (6 points) Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers. Suppose that we have a propositional variable A_n, for every $n \in \mathbb{N}$. We call a variable assignment v an *n-assignment* if

$$v(A_k) = 0\,, \quad \text{for all } k \geq n\,.$$

(There therefore exist exactly 2^n n-assignments.) We call a formula φ an *n-formula* if it only contains implications $\to$ and (some of) the variables $A_0, \dots, A_{n-1}$.

(a) **(1 point)** Find a 3-formula that is true for exactly 7 3-assignments.

(b) **(2 points)** Prove (preferably by induction) that there is no 2-formula that is true for exactly 1 2-assignment.

(c)* **(1 point)** Find a 10-formula that is true for exactly 600 10-assignments.

(d)* **(2 points)** Determine (with proof) for which numbers $n, k \in \mathbb{N}$ there exists an n-formula that is true for exactly k n-assignments.

Solution

(a) for example, $A_2 \to (A_0 \to A_1)$

(b) See the first part of (d) below.

(c) $(A_5 \to ((((A_0 \to A_1) \to A_2) \to A_3) \to A_4)) \to A_6$

(d) There is obviously no 0-formula. Hence, suppose that $n \in \mathbb{N}^+$. First, we prove that every n-formula φ is true for at least 2^{n-1} n-assignments. We proceed by induction on φ.

If $\varphi = A_k$, every n-assignment v with $v(A_k) = 1$ satisfies φ. There are 2^{n-1} such assignments.

Next, suppose that $\varphi = \psi \to \xi$. Then φ is true for every assignment satisfying ξ. By inductive hypothesis, there are at least 2^{n-1} of these.

Since $2^{2-1} = 2$, it follows in particular that there is no 2-formula that is true for exactly 1 2-assignment.

Next we show by induction on n that, for every $2^{n-1} \leq k \leq 2^n$, there is some n-formula that is true for exactly k n-assignments.

For $n = 1$ and $k = 1$, we can take $\varphi = A_0$. For $n = 1$ and $k = 2$, we can take $\varphi = A_0 \to A_0$.

Suppose that $n \geq 2$. We distinguish two cases. If $k < 3 \cdot 2^{n-2}$, we use the inductive hypothesis to find an $(n-1)$-formula φ that is true for $2^n - k$ $(n-1)$-assignments. We consider the formula $\psi := \varphi \to A_{n-1}$. This formula is false if, and only if, φ is true and A_{n-1} is false. There are $2^n - k$ such assignments. Consequently, there are $2^n - (2^n - k) = k$ n-assignments that make ψ true.

Similarly, if $k \geq 3 \cdot 2^{n-2}$, we take an $(n-1)$-formula φ that is true for exactly $k - 2^{n-1}$ $(n-1)$-assignments. We consider the formula $\psi := A_{n-1} \to \varphi$. This formula is false if, and only if, A_{n-1} is true and φ is false. There are $2^{n-1} - (k - 2^{n-1}) = 2^n - k$ such assignments. Consequently, there are $2^n - (2^n - k) = k$ n-assignments that make ψ true.

It follows that there exists an n-formula that is true for at least k n-assignments if, and only if, $n > 0$ and $2^{n-1} \leq k \leq 2^n$.