Homework Sheet 2

Exercise 1 (6 points)

(a) **(3 points)** *Intuition:* We are looking for a binary operation f and a unary operation u that together form a complete system, but such that f together with all other unary operations does not.

Exact question: For $i \in \{0, 1, 2, 3\}$, let u_i be the unary operation mapping $0 \mapsto i \bmod 2$ and $1 \mapsto \lfloor i/2 \rfloor$. Find a binary operation f and an index $i \in \{0, 1, 2, 3\}$ such that the system $\mathcal{L}(f, u_i)$ is complete but the system $\mathcal{L}(f, u_{(i+1) \bmod 4}, u_{(i+2) \bmod 4}, u_{(i+3) \bmod 4})$ is not. Prove your claim.

(b) **(3 points)** *Intuition:* Is there a strongest unary operation?

Exact question: Is there an index $i \in \{0, 1, 2, 3\}$ such that, for every $n \in \mathbb{N}$ and every n-ary operation f, if there is some $j \in \{0, 1, 2, 3\}$ such that $\mathcal{L}(f, u_j)$ is complete, then $\mathcal{L}(f, u_i)$ is also complete? Prove your answer. You can use the statement from the lecture that the system $\mathcal{L}(\neg, \to)$ is complete, but you cannot assume the completeness of other systems.

Exercise 2 (5 points) We consider the logical system $\mathcal{L}(\uparrow, \downarrow, \to)$ where $\uparrow$ and $\downarrow$ are unary and $\to$ is binary. We use the following proof system consisting of three axiom schemata and one rule.

(A1)	$\uparrow\uparrow\downarrow\downarrow\varphi$
(A2)	$\uparrow\downarrow\varphi \to \uparrow\psi$
(A3)	$(\varphi \to \psi) \to (\uparrow\psi \to \uparrow\varphi)$
(MP)	from φ and $\varphi \to \psi$ derive ψ

Determine (with proof) which of the following formulae are derivable in this system.

(a) $\uparrow\uparrow\downarrow A$

(b) $\uparrow\downarrow A$

(c) $\downarrow A$

(d) $\downarrow A \to \downarrow A$