Homework Sheet 3

Exercise 1 (3 points) We consider the vocabulary $\mathcal{L} = \{P, f\}$ (with equality) where P is a unary predicate symbol and f a unary function symbol. Define the following formulae.

$$\begin{aligned}
\varphi &:= \exists x \forall y[P(y) \leftrightarrow y = x]\,, \\
\psi &:= \forall x[P(x) \leftrightarrow f(x) = x]\,, \\
\xi &:= \forall x \exists y[y \neq x \wedge \forall z[f(z) = f(x) \leftrightarrow (z = x \vee z = y)]]\,, \\
\zeta &:= \exists x \neg P(f(f(f(x))))\,.
\end{aligned}$$

For which $n \in \mathbb{N}$ does there exist a structure $\mathcal{M}$ over the vocabulary $\mathcal{L}$ such that $\mathcal{M} \vDash \varphi \wedge \psi \wedge \xi \wedge \zeta$ and such that $\mathcal{M}$ has exactly n elements (no proof necessary)?

Exercise 2 (9 points) We consider the vocabulary $\mathcal{L} = \{P, Q, S\}$ without equality consisting of three relation symbols of arities, respectively, 1, 2, and 2. We call a structure $\mathcal{M}$ over this vocabulary *nice* if it satisfies the following conditions.

- The domain M is the set $2^{\mathbb{N}}$ of all subsets of the set of natural numbers.
- The relation $S_{\mathcal{M}}$ is the proper subset relation: $S_{\mathcal{M}} = \{ \langle A, B \rangle \mid A \subset B \}$.

Find a formula $\varphi(x, y, z)$ over the vocabulary $\mathcal{L}$ such that, given a nice structure $\mathcal{M}$ and a variable assignment e, we have $\mathcal{M} \vDash \varphi[e]$ if, and only if, the following condition holds.[1]

(a) **(1 point)** $e(x) = e(y)$

(b) **(1 point)** $e(z) = e(x) \cap e(y)$

(c) **(1 point)** $e(z) = e(x) \cup e(y)$

(d) **(1 point)** $e(x)$ is the complement of $e(y)$.

Briefly justify the correctness of your answer.

Consider the formulae

$$\begin{aligned}
\psi_Q &:= \forall x \forall y[Q(x, y) \leftrightarrow [S(x, y) \wedge \neg \exists z[S(x, z) \wedge S(z, y)]]]\,, \\
\psi_P &:= \forall x \forall y[Q(x, y) \rightarrow [P(x) \leftrightarrow P(y)]]\,.
\end{aligned}$$

(e) **(1 point)** Note that there exists a unique relation $Q \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $Q = Q_{\mathcal{M}}$, for every nice structure $\mathcal{M}$ satisfying ψ_Q. Describe this relation as explicitly as possible.

(f) **(4 points)** Find as many sets $P \subseteq 2^{\mathbb{N}}$ as possible such that $P = P_{\mathcal{M}}$, for some nice structure $\mathcal{M}$ satisfying $\psi_Q \wedge \psi_P$. Or better, compute exactly how many[2] such sets exist and prove the correctness of your answer.

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[1] In (a) and (d), the variable z does not need to appear in φ.

[2] Here we expect for the answer a cardinal number such as $1, 42, 69, \aleph_0, \aleph_1, 2^{\aleph_0}, \aleph_\omega, 2^{2^{\aleph_{\omega^\omega}}}$.