Homework Sheet 3

Exercise 1 (3 points) We consider the vocabulary $\mathcal{L} = \{P, f\}$ (with equality) where P is a unary predicate symbol and f a unary function symbol. Define the following formulae.

$$\begin{aligned}
\varphi &:= \exists x \forall y [P(y) \leftrightarrow y = x], \\
\psi &:= \forall x [P(x) \leftrightarrow f(x) = x], \\
\xi &:= \forall x \exists y [y \neq x \wedge \forall z [f(z) = f(x) \leftrightarrow (z = x \vee z = y)]], \\
\zeta &:= \exists x \neg P(f(f(f(x)))).
\end{aligned}$$

For which $n \in \mathbb{N}$ does there exist a structure $\mathcal{M}$ over the vocabulary $\mathcal{L}$ such that $\mathcal{M} \vDash \varphi \wedge \psi \wedge \xi \wedge \zeta$ and such that $\mathcal{M}$ has exactly n elements (no proof necessary)?

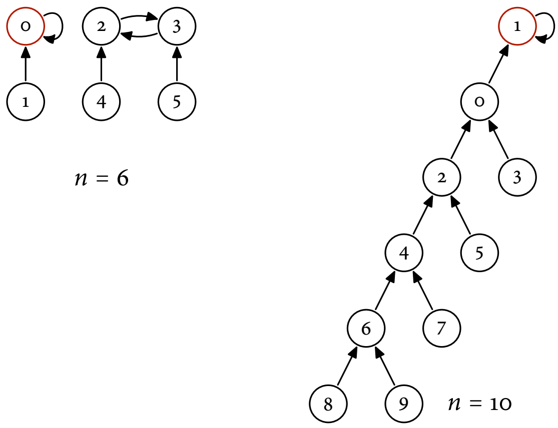
Solution For $n = 6$, we have a model $\mathcal{M}$ with universe $M = \{0, 1, 2, 3, 4, 5\}$, $P_{\mathcal{M}} = \{0\}$, and

$$f_{\mathcal{M}} = \{\langle 0, 0\rangle, \langle 1, 0\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle, \langle 4, 2\rangle, \langle 5, 3\rangle\}.$$

For every even $n \geq 8$, we have a model $\mathcal{M}$ with universe $M = \{0, 1, \dots, n-1\}$, $P_{\mathcal{M}} = \{1\}$, and

$$f_{\mathcal{M}} = \{\langle 0, 1\rangle, \langle 1, 1\rangle\} \cup \{\, \langle 2k+2, 2k\rangle, \langle 2k+3, 2k\rangle \mid k \leq n/2 - 2 \,\}.$$

For other values of n there is no model with n-elements. As an example, let us draw the models for $n = 6$ and $n = 10$. The red circle denotes the element in $P_{\mathcal{M}}$ and the arrows describe the function $f_{\mathcal{M}}$.



Let us explain why models of other sizes do not exist. The formula ξ states that, for every element $a \in M$, there is exactly one other element $b \in M$ with $f(a) = f(b)$. Consequently, every element of M has either exactly two preimages under f or none. If we add the number of these preimages for all element of M, we get the number of elements of M (since f is a function).

It follows that the size of $\mathcal{M}$ is even. Furthermore, by definition of a structure, M cannot be empty. To exclude the remaining cases $n = 2$ and $n = 4$, it is sufficient to prove that the above formulae imply that M has at least 5 elements. By φ, there exists a (unique) element $a \in P_{\mathcal{M}}$. By ψ, we have $f_{\mathcal{M}}(a) = a$, and by ξ a has a second preimage b. BY ζ, there is some element c with $f^3_{\mathcal{M}}(c) \notin P_{\mathcal{M}}$. We have $c \neq a$ and $c \neq b$ since $f_{\mathcal{M}}(a), f_{\mathcal{M}}(b) \in P_{\mathcal{M}}$. Furthermore, $d := f_{\mathcal{M}}(c)$ is different from a and b (since $f_{\mathcal{M}}(d) \notin P_{\mathcal{M}}$) and also from c (by φ). By ξ, d must have a second preimage e under $f_{\mathcal{M}}$. This makes 5 different elements a, b, c, d, e.

Exercise 2 (9 points) We consider the vocabulary $\mathcal{L} = \{P, Q, S\}$ without equality consisting of three relation symbols of arities, respectively, 1, 2, and 2. We call a structure $\mathcal{M}$ over this vocabulary *nice* if it satisfies the following conditions.

- The domain M is the set $2^{\mathbb{N}}$ of all subsets of the set of natural numbers.
- The relation $S_{\mathcal{M}}$ is the proper subset relation: $S_{\mathcal{M}} = \{ \langle A, B\rangle \mid A \subset B \}$.

Find a formula $\varphi(x, y, z)$ over the vocabulary $\mathcal{L}$ such that, given a nice structure $\mathcal{M}$ and a variable assignment e, we have $\mathcal{M} \vDash \varphi[e]$ if, and only if, the following condition holds.[1]

(a) **(1 point)** $e(x) = e(y)$

(b) **(1 point)** $e(z) = e(x) \cap e(y)$

(c) **(1 point)** $e(z) = e(x) \cup e(y)$

(d) **(1 point)** $e(x)$ is the complement of $e(y)$.

Briefly justify the correctness of your answer.

Consider the formulae

$$\psi_Q := \forall x \forall y[Q(x, y) \leftrightarrow [S(x, y) \wedge \neg\exists z[S(x, z) \wedge S(z, y)]]],$$
$$\psi_P := \forall x \forall y[Q(x, y) \rightarrow [P(x) \leftrightarrow P(y)]].$$

(e) **(1 point)** Note that there exists a unique relation $Q \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $Q = Q_{\mathcal{M}}$, for every nice structure $\mathcal{M}$ satisfying ψ_Q. Describe this relation as explicitly as possible.

(f) **(4 points)** Find as many sets $P \subseteq 2^{\mathbb{N}}$ as possible such that $P = P_{\mathcal{M}}$, for some nice structure $\mathcal{M}$ satisfying $\psi_Q \wedge \psi_P$. Or better, compute exactly how many[2] such sets exist and prove the correctness of your answer.

Solution Let $\mathcal{M}$ be a nice structure and e a variable assignment.

(a) A first try would be to take the formula

$$\psi := \forall u[S(u, x) \leftrightarrow S(u, y)].$$

Clearly, ψ is true if $e(x) = e(y)$. Conversely, suppose that $e(x) \neq e(y)$. Then there is some $n \in \mathbb{N}$ with $n \in e(x)$ and $n \notin e(y)$ (or the other way round). Hence, $\{n\} \subseteq e(x)$, but $\{n\} \nsubseteq e(y)$. If $e(x) \neq \{n\}$, it follows that

$$\mathcal{M} \nvDash \psi[e].$$

But if $e(x) = \{n\}$, the only proper subset of $e(x)$ is $\varnothing$. So, if $e(y) = \{k\}$ for $k \neq n$, the only proper subset of $e(y)$ is also $\varnothing$ and it follows that

$$\mathcal{M} \vDash \psi[e].$$

Consequently, the formula ψ above does not quite work.

If we take the dual formula

$$\xi := \forall u[S(x, u) \leftrightarrow S(y, u)]$$

[1] In (a) and (d), the variable z does not need to appear in φ.

[2] Here we expect for the answer a cardinal number such as $1, 42, 69, \aleph_0, \aleph_1, 2^{\aleph_0}, \aleph_\omega, 2^{2^{\aleph_{\omega^\omega}}}$.

instead, we have a similar problem that ξ holds if $e(x) = \mathbb{N} \setminus \{n\}$ and $e(y) = \mathbb{N} \setminus \{k\}$, for $n \neq k$.

Since the cases where the two formulae ψ and ξ fail are disjoint, we can combine these formulae to get

$$\varphi_{\mathrm{a}} := \psi \wedge \xi\,.$$

(b) Note that the intersection is the infimum with respect to the inclusion ordering. Hence, we can set

$$\varphi_{\mathrm{b}} := z \subseteq x \wedge z \subseteq y \wedge \forall t[(t \subseteq x \wedge t \subseteq y) \rightarrow t \subseteq z]\,.$$

(c) Dually to (b), we can use

$$\varphi_{\mathrm{c}} := x \subseteq z \wedge y \subseteq z \wedge \forall t[(x \subseteq t \wedge y \subseteq t) \rightarrow z \subseteq t]\,.$$

(d) It is sufficient to state that the sets x and y are disjoint and that their union is all of $\mathbb{N}$. Guessing the sets $t := \varnothing$ and $u := \mathbb{N}$ and using the formulae from (b) and (c), we can write

$$\varphi_{\mathrm{d}} := \exists t \exists u[\forall v(t \subseteq v \wedge v \subseteq u) \wedge \varphi_{\mathrm{b}}(x, y, t) \wedge \varphi_{\mathrm{c}}(x, y, z)]\,.$$

A different solution to (b)–(d) works with singleton sets. The formula $J(x) := \exists y \forall z[S(z, x) \leftrightarrow \varphi_{\mathrm{a}}(z, y)]$ states that x is a singleton.

$$\begin{aligned}
\varphi_{\mathrm{b}} &:= \forall t[J(t) \rightarrow [t \subseteq z \leftrightarrow (t \subseteq x \wedge t \subseteq y)]]\,,\\
\varphi_{\mathrm{c}} &:= \forall t[J(t) \rightarrow [t \subseteq z \leftrightarrow (t \subseteq x \vee t \subseteq y)]]\,,\\
\varphi_{\mathrm{d}} &:= \forall t[J(t) \rightarrow \neg(t \subseteq x \leftrightarrow t \subseteq y)]\,.
\end{aligned}$$

(e) Clearly, Q must include all pairs $\langle A, B\rangle$ such that $A \subset B$ and $|B \setminus A| = 1$. Conversely, if $A \subset B$ and $B \setminus A$ contains more than one element, we have $A \subset A \cup \{n\} \subset B$, for any $n \in B \setminus A$. Hence, $\langle A, B\rangle$ does not belong to Q. Therefore,

$$Q = \{\, \langle A, A \cup \{n\}\rangle \mid A \subset \mathbb{N}, n \in \mathbb{N} \setminus A \,\}\,.$$

(f) Clearly, the equivalence $P(x) \leftrightarrow P(y)$ always holds if P is true for all sets, or if it is true for no set. So we have at least these 2 choices for P.

The formula ψ_P only requires that $P(x) \leftrightarrow P(y)$ holds for all $\langle x, y\rangle \in Q$, that is, for all pairs where y contains exactly one more element that x. By induction on the difference $y \setminus x$ it follows that $P(x) \leftrightarrow P(y)$ must hold for all pairs that differ by a finite number of elements. This condition is also sufficient for the validity of φ_P. Thus, we can choose P to be true for all finite sets and false for all infinite ones, or vice versa. This gives already 4 choices.

Next we can distinguish between infinite sets whose complement is finite and those where the complement is infinite. This gives a total of $2^3 = 8$ choices.

For the general statement, consider the relation

$$E := \{\, \langle A, B\rangle \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid A \oplus B \text{ is finite} \,\}\,.$$

($\oplus$ denotes the symmetric difference.) Note that E is an equivalence relation: reflexivity holds since $A \oplus A = \varnothing$ is finite; symmetry holds since $A \oplus B = B \oplus A$; and transitivity holds since, if $A \oplus B$ and

$B \oplus C$ are finite, then $A \oplus C = (A \oplus B) \oplus (B \oplus C)$ is the symmetric difference of two finite sets and, therefore, also finite.

Let $R := 2^{\mathbb{N}}/E$ be the quotient. We have argued above that the predicate P satisfies our formula if, and only if, it respects E, i.e., if, and only if, it either contains all elements of a given E-class, or none of them. This gives $2^{|R|}$ choices for P.

To conclude our argument, we show that $|R| = 2^{\aleph_0}$ (which means that there are $2^{2^{\aleph_0}}$ possible choices for P). Clearly,

$$|R| \leq 2^{|\mathbb{N}|} = 2^{\aleph_0}.$$

For the converse inequality, we find an injection $2^{\mathbb{N}} \to R$, i.e., a function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $f(A)$ and $f(B)$ are not E-equivalent for $A \neq B$. To do so, we fix an injection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. For instance, we can set

$$g(a,b) := p_a^{b+1}, \quad \text{where } p_0, p_1, \dots \text{ is an enumeration of all prime numbers.}$$

Then we can set

$$f(A) := \{ g(a,b) \mid a \in A,\ b \in \mathbb{N} \}.$$

(Intuitively, for each $a \in A$, we include in $f(A)$ the entire row of the table for g (see below).)

If $A \neq B$, then $f(A) \oplus f(B)$ contains all numbers $g(a,b)$ with $a \in A \oplus B$ and $b \in \mathbb{N}$. In particular, the symmetric difference is infinite.

g(a,b)	b = 0	b = 1	b = 2	b = 3	b = 4	b = 5	⋯
a = 0	2	4	8	16	32	64	⋯
a = 1	3	9	27	81	243	729	⋯
a = 2	5	25	125	625	3125	15625	⋯
a = 3	7	49	343	2401	18087	117649	⋯
a = 4	11	121	1331	14641	161051	1771561	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	