Homework Sheet 4

Exercise 1 (2 points) We consider the vocabulary $\mathcal{L} = \{P\}$ without equality, where P is a binary relation symbol. Let T be the following theory over $\mathcal{L}$.

$$T := \{\forall x \exists y P(x, y),\ \forall x \exists y \neg P(y, x)\}.$$

Determine (with proof) whether the theory T is complete.

Exercise 2 (4 points) We consider the vocabulary $\mathcal{L} = \{\sim, f, c\}$ with equality, where $\sim$ is a binary relation symbol and f, c are function symbols of arity, respectively, 1 and 0. Let T be the following theory over $\mathcal{L}$.

$$T := \{x \sim f(x),\ f^4(c) = c\}.$$

Write down the canonical structure $\mathcal{M}$ for T. Show that $\sim$ really is the relation you claim it is.

Exercise 3 (7 points) We consider the vocabularies $\mathcal{L} = \{Q\}, \mathcal{L}_1 = \{Q, P\}, \mathcal{L}_2 = \{Q, f\}$ with equality, where Q is a binary relation symbol, P is a unary relation symbol, and f a unary function symbol. We are given the following formulae over $\mathcal{L}$ (for an arbitrary $n \in \mathbb{N}$).

$$\vartheta \equiv \forall x \forall y [Q(x, x) \wedge [Q(x, y) \leftrightarrow Q(y, x)]]$$

$$\varphi_n \equiv \forall y \exists x_1 \cdots \exists x_n \bigwedge_{i=1}^{n} \Big[Q(y, x_i) \wedge \bigwedge_{j=i+1}^{n} x_i \neq x_j\Big]$$

$$\psi_n \equiv \exists x_1 \cdots \exists x_n \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} \big[x_i \neq x_j \wedge Q(x_i, x_j)\big]$$

$$\xi_n \equiv \forall x_0 \exists x_1 \cdots \exists x_n \bigwedge_{i=0}^{n} \bigwedge_{j=i+1}^{n} \big[x_i \neq x_j \wedge Q(x_i, x_j)\big]$$

Let $\mathcal{M}$ be an structure over the vocabulary $\mathcal{L}$ with universe M. We call a subset $A \subseteq M$ a *clique* if $A \times A \subseteq Q_{\mathcal{M}}$. A structure $\mathcal{M}$ is *nice* if there is an infinite clique $A \subseteq M$. We call a theory T over $\mathcal{L}$ *good* if every model $\mathcal{M}$ of T is nice.

Given a structure $\mathcal{M}$ over a vocabulary $\mathcal{L}' \supseteq \mathcal{L}$, we denote by $\mathcal{M}|_{\mathcal{L}}$ its $\mathcal{L}$-reduct, i.e., we forget all relations and function not in $\mathcal{L}$. We call a theory T over a vocabulary $\mathcal{L}' \supseteq \mathcal{L}$ *great* if a structure $\mathcal{M}$ over $\mathcal{L}$ is nice if, and only if, it can be extended to a model of T, i.e., there exists a model $\mathcal{M}'$ of T such that $\mathcal{M}'|_{\mathcal{L}} = \mathcal{M}$.

(a) **(1 point)** Show that the theory $R := \{\, \vartheta, \varphi_n \mid n \in \mathbb{N} \,\}$ is not good.

(b) **(1 point)** Show that the theory $S := \{\, \vartheta, \psi_n \mid n \in \mathbb{N} \,\}$ is not good.

(c) **(1.5 points)** Give an example of a great theory T over the vocabulary $\mathcal{L}_1$. Briefly explain why your example is correct.

(d) **(1.5 points)** Give an example of a finite great theory U over the vocabulary $\mathcal{L}_2$. Briefly explain why your example is correct.

(e) **(2 points)** Determine (with proof) whether the theory $V := \{\, \vartheta, \xi_n \mid n \in \mathbb{N} \,\}$ is good.