MA010 Graph Theory: Lecture 3

We start with a characterization of planar graphs. A *subdivision* of a graph G is a graph obtained from G by repeatedly subdividing some of its edges, i.e., replacing some of its edges with paths, which stands for removing an edge and identifying its end vertices with the first and the last vertices of a path. Observe that if G is a graph and G' is a subdivision of G, then G is planar if and only if G' is planar. In particular, if G contains a subdivision of K_5 as a subgraph, then G is not planar.

Theorem (Kuratowski's Theorem)**.** *A graph G is planar if and only if G does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Proposition (Euler's Formula)**.** *Every connected plane graph G satisfies that*

$$n + f = m + 2 \tag{1}$$

where n, m and f are the numbers of vertices, edges and faces of G, respectively.

Proof. We proceed by induction on the number of edges of G. The base of the induction is $m = 0$, when G consists of a single vertex. Hence, $n = 1$ and $f = 1$, and the identity (1) holds.

We now present the induction step. Let G be a plane graph with $m \geq 1$ edges, and let n and f be the numbers of vertices and faces of G, respectively. We distinguish two cases based on whether G has an edge e whose removal keeps G connected or G has no such edge.

We start with the former case. Let ww' be an edge such that removal of ww' keeps the graph G connected, and let G' be the connected plane graph obtained from G by deleting the edge ww'. The graph G' is connected and so there is a path P from w to w'; let C be the cycle consisting of the edge ww' and the path P. Note that the edge ww' is incident with two different faces of G: one is contained inside the area bounded by the cycle C and the other outside. Hence, the number of faces of the graph G' is $f - 1$ as the two faces incident with ww' gets merged to a single face by removal of the edge ww'. Since the number of vertices of G' is n and the number of edges is $m - 1$, the induction assumption implies that $n + (f - 1) = (m - 1) + 2$, which yields that (1) holds for G.

In the latter case, G is a tree (as we have proven in Lecture 3). Since G has at least two vertices, it has a leaf and let v be any of its leaves. Let G' be the graph obtained from G by removing the leaf v (together with the incident edge). Observe that the number of vertices of G' is $n - 1$, the number of edges is $m - 1$ and the number of faces is f. The induction assumption implies that $(n - 1) + f = (m - 1) + 2$, which yields that (1) holds for G. □

Proposition. *Every planar graph with $n \geq 3$ has at most $3n-6$ edges.*

Proof. Let G be a planar graph and G' be a triangulation that contains G as a spanning subgraph. Let n, m and f be the number of vertices, edges and faces of G', respectively. Since each face is bounded by three edges and each edge is incident with two faces, the number of incidences of edges and faces is $3f = 2m$. It follows that $f = 2m/3$. Euler's Formula now yields that $n + 2m/3 = m + 2$, which implies that $m = 3n-6$. Hence, the number of edges of G is at most $m = 3n-6$. $\square$

Note that the just proven proposition implies that K_5 is not planar as K_5 has 5 vertices and 10 edges, which is larger than $3 \cdot 5 - 6 = 9$.

Proposition. *Every planar graph has a vertex of degree at most five.*

Proof. Let G be a planar graph and let n be its number of vertices. If $n \leq 6$, then the graph G trivially contains a vertex of degree at most five. Suppose that $n \geq 7$ and the degree of each vertex of G is at least six. Since the sum of the degrees of G, which is at least $6n$, is equal to twice the number of edges (see the proof of the Hand-shaking Lemma), the number of edges of G is at least $3n$. However, this is impossible as G can have at most $3n-6$ edges. $\square$

A *proper k-vertex-coloring* of a graph G is a function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge uv of G. We will often simply say a *coloring* or *k-coloring* instead of a proper k-vertex-coloring unless we want to emphasize that the end vertices of each edge have different colors, that we color the vertices or highlight the number of colors. The *chromatic number* of a graph G is the smallest k such that G has a proper k-vertex-coloring; the chromatic number of G is denoted by $\chi(G)$.

One of the most well-known results in graph theory is the Four Color Theorem, which can be traced back to Francis Guthrie to 1852. The computer assisted proof was given by Appel and Haken in 1976, and a simplified and refined (computer assisted) proof by Robertson, Seymour and Thomas in 1996; Gonthier and Werner formalized the argument in the Coq prover in 2005.

Theorem (Four Color Theorem)**.** *The chromatic number of every planar graph is at most four.*

We will prove the following weaker result.

Theorem (Five Color Theorem)**.** *The chromatic number of every planar graph is at most five.*

Proof. The proof proceeds by induction on the number of vertices of a planar graph G. The base case is when G has a single vertex when $\chi(G) = 1$. We now present the induction step. Let G be an n-vertex planar graph with $n \geq 2$.

Suppose that G has a vertex of degree at most four and let w be such a vertex. Consider the $(n-1)$-vertex graph G' obtained from G by removing the vertex w. By induction, the vertices of G' can be colored using five colors. Since this coloring can be extended to the whole graph G by assigning w a color different from the colors of the neighbors of w, the graph G can also be colored using five colors.

We assume in the rest of the proof that the minimum degree of G is at least five. As we have shown in previous lectures, G is a spanning subgraph of a triangulation, say G_T, and G_T has a vertex of degree at most five, say w. Since the minimum degree of G is at least five, the degree of w is actually equal to five. Let $v_1, \ldots, v_5$ be the neighbors of w listed in the cyclic order around w. Observe that both v_1v_3 and v_2v_4 cannot be edges of G_T (because of the planarity); by symmetry, we will assume that v_1v_3 is not an edge.

Let G' be the $(n-2)$-vertex graph obtained from G_T as follows: Remove the vertices w, v_1 and v_3, introduce a new vertex w' drawn where w was, join each neighbor u of v_1 to w' by drawing an edge along the edges wv_1 and v_1u, and similarly join each neighbor u of v_3 to w' by drawing an edge along the edges wv_3 and v_3u (if a vertex u is a neighbor of both v_1 and v_3, include only one of the edges). By induction, the vertices of the graph G' can be colored using five colors. We now define a coloring of the vertices of G that uses five colors: all vertices different from w, v_1 and v_3 get the same color that they have in G', and the vertices v_1 and v_3 get the color of the vertex w'. Finally, the vertex w gets a color that is not used on any of its neighbors; note that there are at most four colors used on the neighbors of w as the vertices v_1 and v_3 have the same color. Hence, it is possible to assign w one of the five colors and so we have obtained a coloring of the vertices of G with five colors. This completes the proof of the induction step and so the proof of the theorem. □

We conclude with observing that

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

where $\omega(G)$ is the *clique number* of G, which is the order of the largest complete subgraph of G, and $\Delta(G)$ is the maximum degree of a vertex of G. The first inequality is trivial as vertices of any complete subgraph must get mutually distinct colors in any proper coloring. To establish the inequality $\chi(G) \leq \Delta(G)+1$, fix a graph G and consider vertices $v_1, \ldots, v_n$ of G listed in any order. We now color the vertices in the order $v_1, \ldots, v_n$: when the vertex v_i is to be colored, there are at most $\deg_G(v_i) \leq \Delta(G)$ colors that are already assigned to its neighbors, and so one of the $\Delta(G) + 1$ colors can be assigned to v_i without creating a monochromatic edge.

Note that the inequality $\chi(G) \leq \Delta(G) + 1$ is tight for odd cycles and complete graphs, which we will discuss in more detail in the next lecture, and the difference can be arbitrarily large as witnessed by complete bipartite graphs.