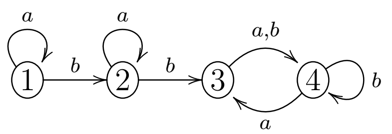
Algebra I – autumn 2023 – written exam template

All your assertions should be carefully justified.

1. **(10 points)** Decide whether ... is a semigroup/monoid/group/ring/integral domain/field.
   (*for instance,* Decide whether $(\mathbb{Z}, *)$, where $*$ is the operation defined by the rule $a * b = a + b - ab$ for all $a, b \in \mathbb{Z}$, is a semigroup and whether it is a group.)
   or
   Decide whether ... is a subsemigroup/submonoid/subgroup/normal subgroup/ subring/ideal of ... .

2. **(10 points)** Determine all elements of the transition monoid of the automaton ... .
   (*the automaton can be, for instance,*



)

3. **(15 points)** Find a direct product of well-known groups that is isomorphic to the quotient group $(G, \cdot)/H$.
   for instance,

$$(G,\cdot) = \left(\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ b & c & 1 \end{pmatrix} \mid a \in \mathbb{Q}\setminus\{0\},\ b \in \mathbb{C},\ c \in \mathbb{R} \right\}, \cdot\right)$$

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ b\mathrm{i} & c & 1 \end{pmatrix} \mid a \in \{-1,1\},\ b, c \in \mathbb{R} \right\}$$

4. **(10 points)** Find the minimal polynomial of the number ... over $\mathbb{Q}$.
   (*the number can be, for instance,* $1 + \sqrt{\sqrt[3]{2} - 1} \cdot \mathrm{i}$, $\sqrt{3} + \sqrt[3]{\sqrt{3} + 3}$, $\sqrt[3]{9} - \sqrt[3]{3} + 3$)

5. **(15 points)** Express the number $\frac{1}{\cdots}$ without using other than rational numbers in denominators.
   (*the number can be, for instance,* $\frac{1}{\alpha^2 - \alpha + 1}$, where α satisfies $\alpha^3 + 2\alpha^2 + 2\alpha = -2$)

6. **– 7. (2 × 10 points)** Provide an example of a semigroup/group/ring/homomorphism with given properties.
   (*for instance,* a group that contains elements of every possible order *or* an infinite group and its subgroup of index 10)

8. **(5 points)** Define ... .

9. (5 points) Formulate the theorem . . . .

10. (10 points) Prove . . . .

The answer to each of the questions 8. – 10. was presented during the lectures.