PA170 Digital Geometry

Lecture 01 – Introduction + Grids and Adjacency

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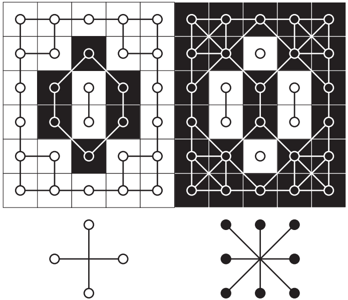
INTRODUCTION

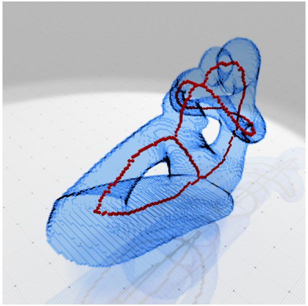
About PA170

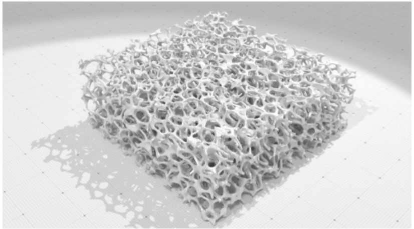
- Lectures (Monday 14:00 – 15:40, B204)
  - Introduction of the basic terms and necessary theory
- Exercises (Monday 16:00 – 16:50, B204)
  - Deeper understanding and application of the introduced terms and theory in practice
- Homeworks
  - Irregular and voluntary, assigned at exercises
  - Awarded by extra points (up to 10 points)
- Examination (one term before Christmas, other terms in January 2024)
  - Written (mandatory) and oral (voluntary) parts (up to 100 points)
  - Grading scheme
    - A: at least 91 points
    - B: 90 – 81 points
    - C: 80 – 71 points
    - D: 70 – 61 points
    - E: 60 – 51 points
    - F: less than 51 points
- Recommended literature
  - R. Klette & A. Rosenfeld, *Digital Geometry: Geometric Methods for Digital Picture Analysis*, Elsevier 2004 (V238 in the library at FI MU)

Digital Geometry in a Nutshell

- It emerged in the second half of the 20th century
- Its mathematical roots are in graph theory and discrete topology
- It focuses on geometric or topologic properties of discrete sets
- It often attempts to obtain quantitative information about objects by analyzing digitized image data in which the objects are represented by such sets







Digital Images

- Digital images are obtained by digitizing continuous functions

256×256 pixels | 128×128 pixels | 64×64 pixels | 32×32 pixels

Sampling
(domain discretization)

Quantization
(co-domain discretization)

256 levels | 32 levels | 8 levels | 2 levels

- A digital image I is a discrete function, defined on a finite, regular, orthogonal grid $\mathbb{G}$, which assigns a value $I(p)$ from a finite set of values $\mathbb{V}$ to each image element $p \in \mathbb{G}$

Common Types of Digital Images

Binary Images ($\mathbb{V} = \{0, 1\}$)

- The image elements with the assigned values of 1 are black and form foreground $\langle I \rangle$
- The image elements with the assigned values of 0 are white and form background $\overline{\langle I \rangle}$

Grayscale Images ($\mathbb{V} = \{0, \ldots, G_{\max}\}$)

- The elements of $\mathbb{V}$ are called intensities
- The image elements with higher intensities correspond to lighter gray levels
- The image elements with zero intensity are black and form background

Floating-Point Images (e.g., Euclidean distance maps)

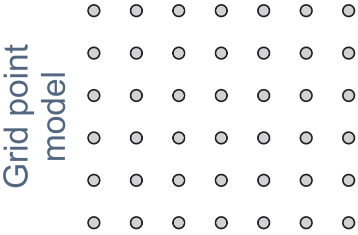
Color Images (e.g., RGB or HSV)

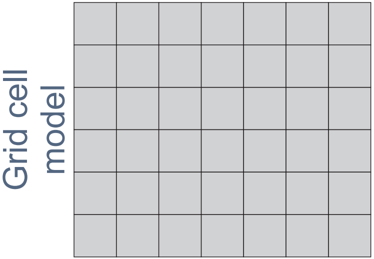
Multi-Channel Images (e.g., microscopy or satellite images)

Tensor Images (e.g., DT-MRI)

GRIDS AND ADJACENCY

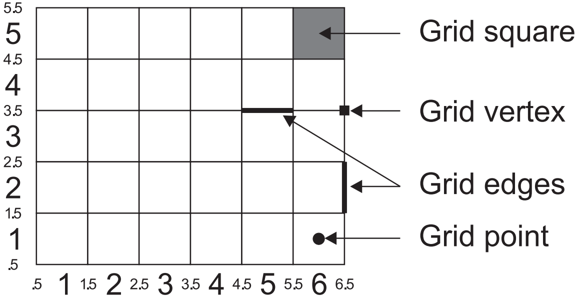
Basic Grid Models





Elements of a Grid

- A grid point is an element of $\mathbb{Z}^2$ or $\mathbb{Z}^3$
- A grid vertex (0-cell) is shifted by 0.5 in each direction from a grid point
- A grid edge (1-cell) joins two grid vertices at Euclidean distance of 1 from each other
- A grid square (2-cell) is defined by four grid edges that form a square
- A grid cube (3-cell) is defined by six grid squares that form a cube in 3D



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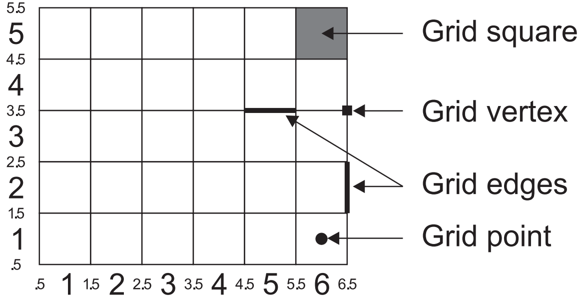
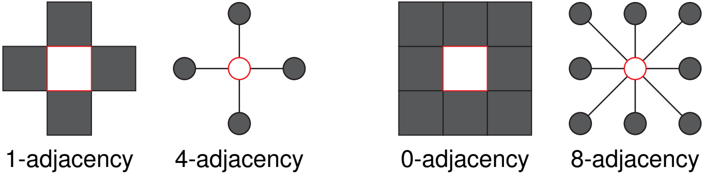


Pixel (2D grids) = 2D grid point or 2-cell

Voxel (3D grids) = 3D grid point or 3-cell

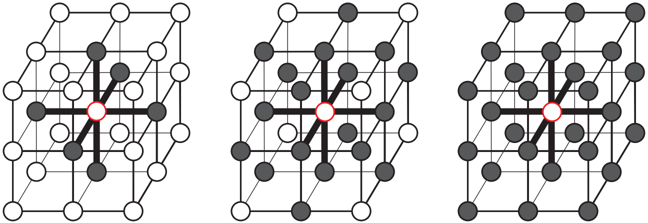
α-Adjacency (A_α) on 2D Grids

- Two 2-cells c_1 and c_2 are called α-adjacent iff $c_1 \neq c_2$ and their intersection contains an α-cell ($\alpha \in \{0, 1\}$)
- Two 2D grid points p_1 and p_2 are called
  - 4-adjacent iff $\|p_2 - p_1\|_1 = 1$
  - 8-adjacent iff $\|p_2 - p_1\|_\infty = 1$
- The grid $\mathbb{G}_{m,n}$ defined by $m \times n$ 2-cells and A_0 or A_1 relation is isomorphic to the grid defined by $m \times n$ 2D grid points and A_8 or A_4 relation, respectively



α-Adjacency (A_α) on 3D Grids

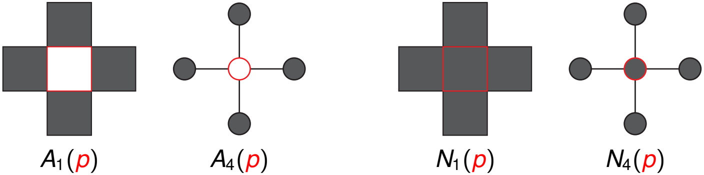
- Two 3-cells c_1 and c_2 are called α-adjacent iff $c_1 \neq c_2$ and their intersection contains an α-cell ($\alpha \in \{0, 1, 2\}$)
- Two 3D grid points p_1 and p_2 are called
  - 6-adjacent iff $0 < \|p_2 - p_1\|_2 \leq 1$
  - 18-adjacent iff $0 < \|p_2 - p_1\|_2 \leq \sqrt{2}$
  - 26-adjacent iff $0 < \|p_2 - p_1\|_2 \leq \sqrt{3}$
- The grid $\mathbb{G}_{l,m,n}$ defined by $l \times m \times n$ 3-cells and A_2, A_1, or A_0 relation is isomorphic to the grid defined by $l \times m \times n$ 3D grid points and A_6, A_{18}, or A_{26} relation, respectively



6-adjacency 18-adjacency 26-adjacency

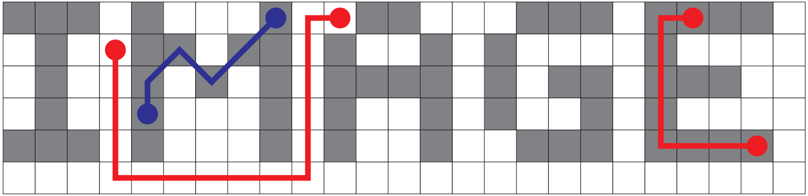
Adjacency Set and Neighborhood

- Let A be a symmetric and irreflexive adjacency relation on a set S
- $A(p) = \{q : q \in S \wedge qAp\}$ is called the adjacency set of $p \in S$
- $N(p) = A(p) \cup \{p\}$ is called the (smallest nontrivial) neighborhood of $p \in S$ defined by the adjacency relation A
- N defines a symmetric and reflexive relation on S



Connectedness, Paths, and Components

- The reflexive and transitive closure of an adjacency relation on a set S defines a connectedness relation on S
- A path that joins $p \in S$ and $q \in S$ is a sequence $p_0, \ldots, p_n$ of elements in S such that $p_0 = p$, $p_n = q$, and p_i is adjacent to p_{i-1} $(1 \leq i \leq n)$
- A set $M \subseteq S$ is called connected iff all $p, q \in M$ are joined by a path in M
- Maximal connected subsets of S are called (connected) components of S



Examples of two 4-paths and one 8-path

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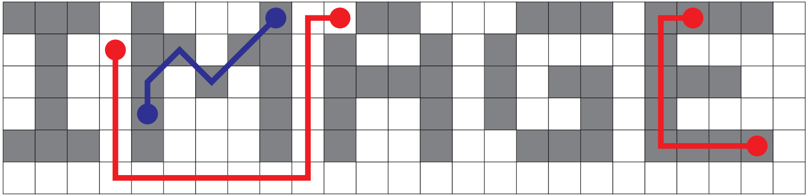
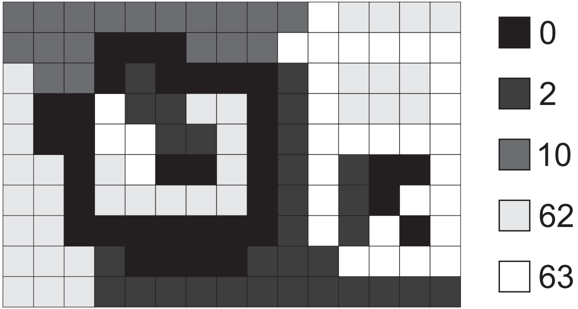


Examples of two 4-paths and one 8-path

What is the number of 4-connected and 8-connected components?

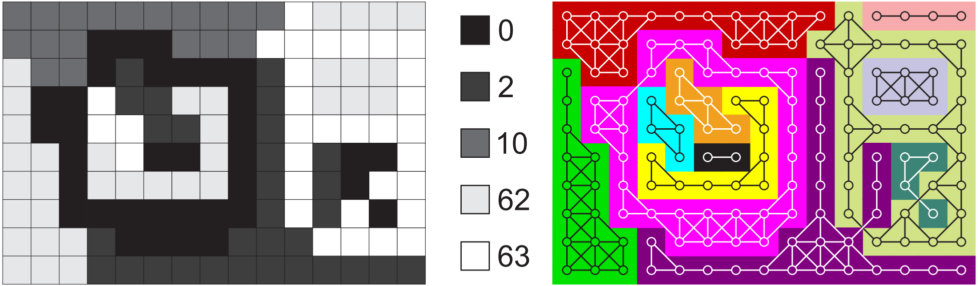
Adjacency on Images

- The number of image elements α-adjacent to an image element is (almost) always constant over the whole grid
- That number can vary for adjacency relations defined on images because not only locations but also values of image elements are taken into consideration



(I, α)-Adjacency

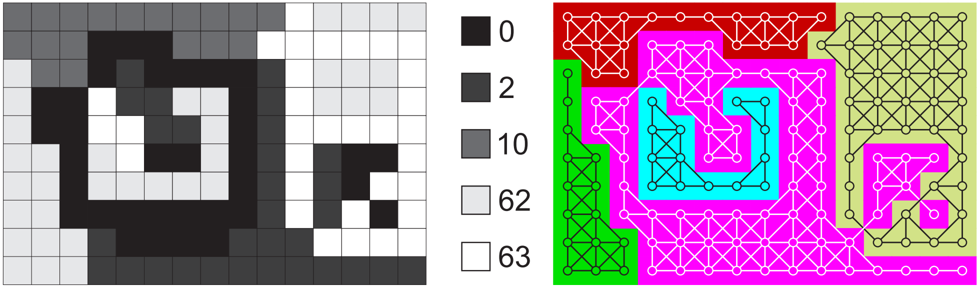
- Two image elements p and q of an image I are called I-equivalent iff $I(p) = I(q)$
- Two image elements p and q of an image I are called (I, α)-adjacent iff they are I-equivalent and α-adjacent
- (I, α)-adjacency may lead to crossing difficulties (topological conflicts)



Connected components for $(I, 8)$-adjacency or $(I, 0)$-adjacency

(σ, α)-Adjacency

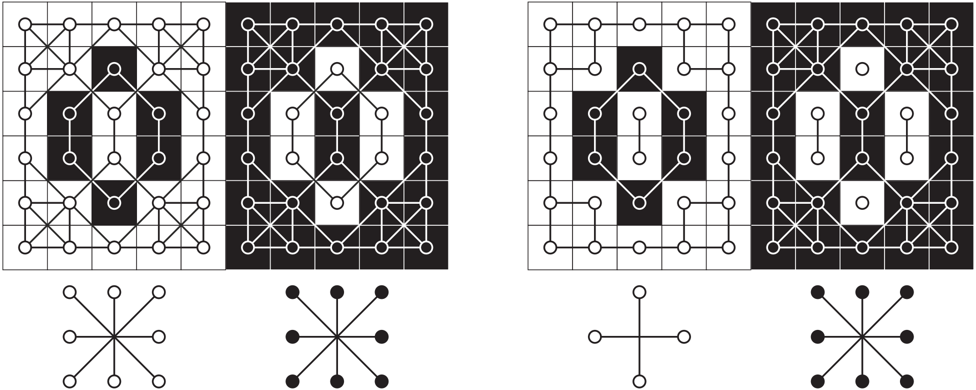
- *"Uncertainties"* in image values can be modeled by a similarity relation σ on $\mathbb{V}$, which is reflexive and symmetric
- Two image elements p and q of an image I are called (σ, α)-adjacent iff $pA_\alpha q$ and $I(p)\sigma I(q)$
- (σ, α)-adjacency generalizes (I, α)-adjacency and may lead to crossing difficulties



Connected components for $(\sigma, 8)$-adjacency or $(\sigma, 0)$-adjacency ($p\sigma q \Leftrightarrow |I(p) - I(q)| \leq 2$)

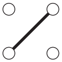
(α_1, α_2)-Adjacency

- (α_1, α_2)-adjacency avoids topological conflicts in binary images
- It considers (I, α_1)-adjacency for foreground and (I, α_2)-adjacency for background
- Commonly used, topologically compatible α-adjacency pairs:
  - A_8 and A_4 (or A_0 and A_1) in 2D
  - A_{26} and A_6 (or A_0 and A_2), and A_{18} and A_6 (or A_1 and A_2) in 3D

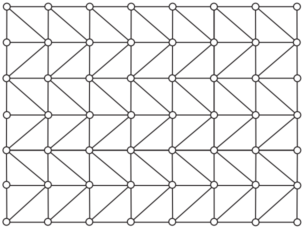


Switch Adjacency (A_s, s-Adjacency)

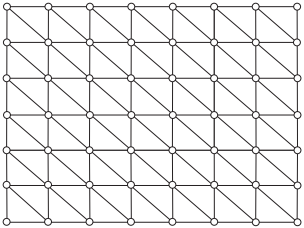
- Switch adjacency avoids topological conflicts in 2D images
- A relation A_s on a set of 2D grid points is called switch adjacency iff it contains 4-adjacency (i.e., $A_s \supseteq A_4$) and exactly one of the two diagonally adjacent pixels in each 2×2 block of pixels:

Switch to the left  Switch to the right 

- The states of the switches can be driven by locations and/or pixel values



Regular switching



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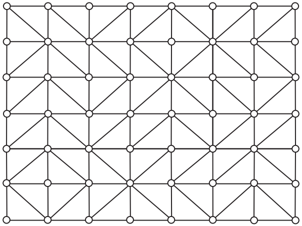
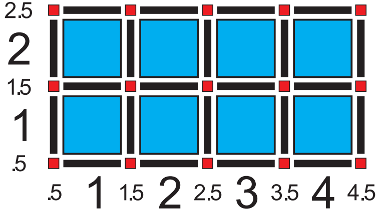


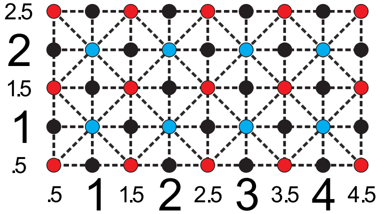
Irregular switching

Incidence Relation

- Two sets are called incident iff one of them contains the other (i.e., any set is incident with itself)
- A 3D grid vertex (0-cell) is incident with six grid edges (1-cells); a grid square (2-cell) is incident with four grid edges; and a grid cube (3-cell) is incident with 12 grid edges
- The incidence relation between all the cells defines grid cell incidence model (incidence grid)

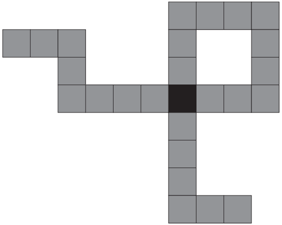


0-, 1-, and 2-cells

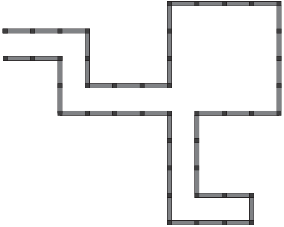


Incidence graph

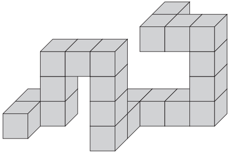
Paths in Incidence Grids



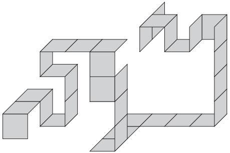
1-path of 2-cells



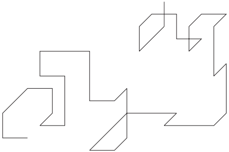
0-path of 1-cells



2-path of 3-cells

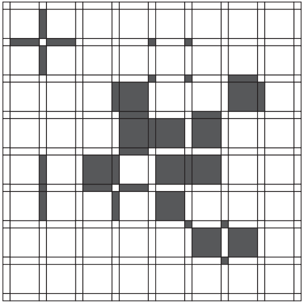


1-path of 2-cells



0-path of 1-cells

- A subset M of an incidence grid is called complete iff, for any cell c such that all cells incident with c and of higher dimensions than c are in M, we also have $c \in M$



Add the minimum number of cells to make the given subset of an incidence grid complete

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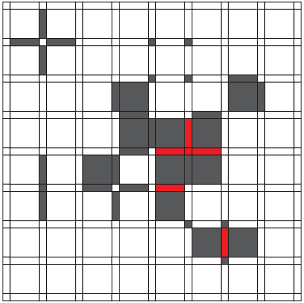

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Summary: Commonly Used Models in Digital Geometry

Grid	**Adjacency**	**Graph**
Points →	$\alpha \in \{4,8,6,18,26\}$, (α_1,α_2), (σ,α), s →	Adjacency graph
Cells →	$\alpha \in \{0,1,2,3\}$, (α_1,α_2), (σ,α) →	Adjacency graph
Cells →	Incidence →	Incidence graph