PA170 Digital Geometry

Lecture 02: Digitization

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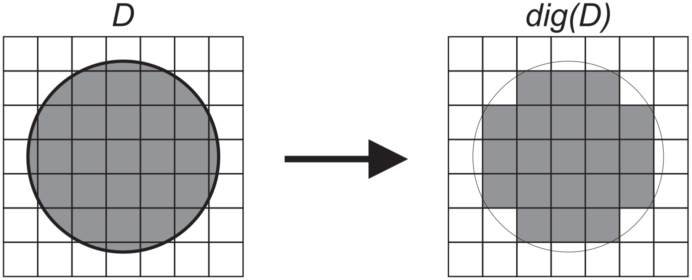
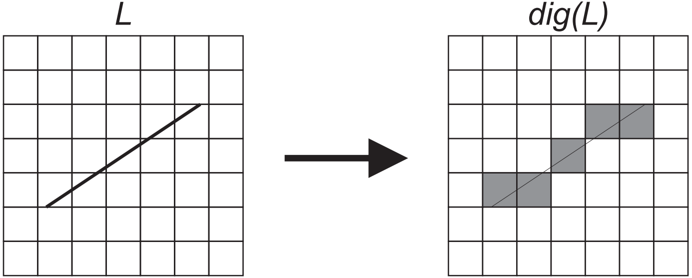
Motivation: Transformation of a Continuous Set to a Discrete Set

Digitization of a real disk D

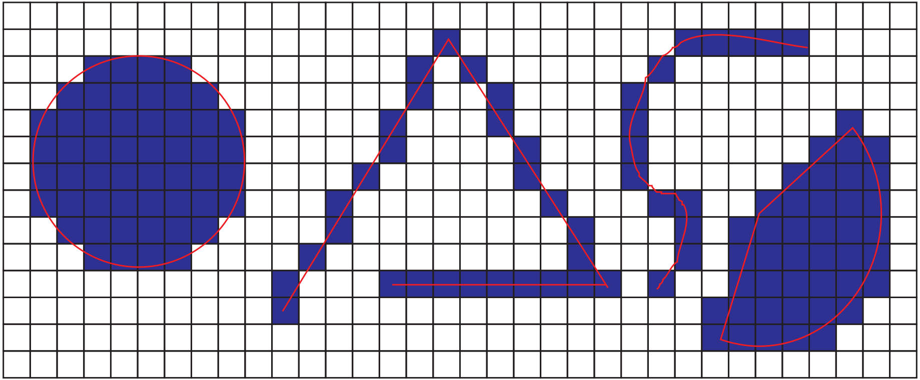


Digitization of a real line L



Digital Geometric Figures

- A connected set of grid points is called a digital geometric figure (e.g., digital line, digital square, digital disk, or digital sphere), if there exists a (continuous) geometric figure of the same kind, which has that set as its digitization



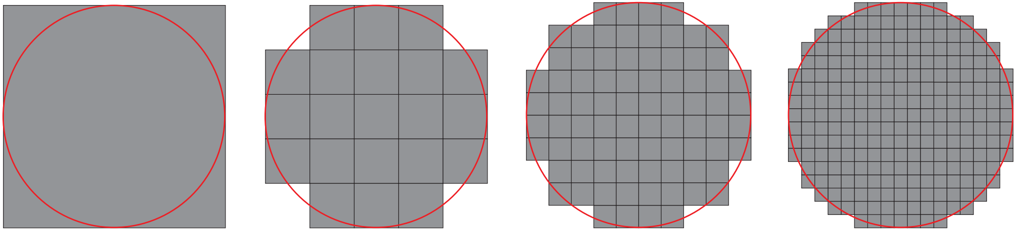
Example: Convergence of Estimators

- A real disk D of unit diameter has the area $\mathcal{A}(D) = \frac{\pi}{4}$ and the perimeter $\mathcal{P}(D) = \pi$
- The area of a digitized disk converges toward the area of the real disk with an increasing grid resolution h:

$$\lim_{h \to \infty} \mathcal{A}(dig_h(D)) = \mathcal{A}(D) = \frac{\pi}{4}$$

- The perimeter of a digitized disk does not converge toward the perimeter of the real disk:

$$\lim_{h \to \infty} \mathcal{P}(dig_h(D)) = 4$$



$h = 1$ $h = 5$ $h = 10$ $h = 17$

DIGITIZATION MODELS

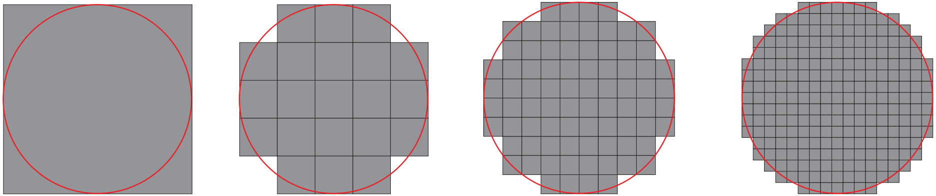
Look into History

C.F. Gauss (1777 – 1855)

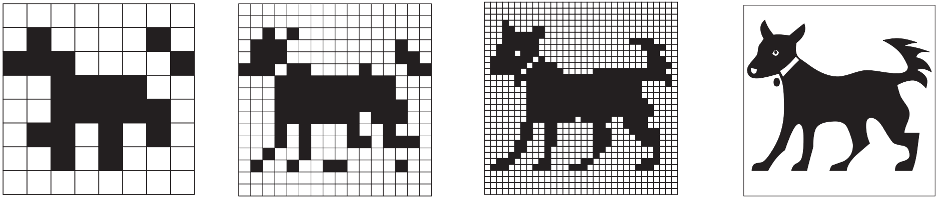
C. Jordan (1838 – 1922)

Source: https://mathshistory.st-andrews.ac.uk/

- Gauss studied the measurement of the area of a planar set $S \subset \mathbb{R}^2$ by counting the grid points $(i, j) \in \mathbb{Z}^2$ contained in S
- The Gauss digitization $G_h(S)$ of a planar set S on a 2D grid of resolution h is the union of the grid squares (2-cells) with center points in S



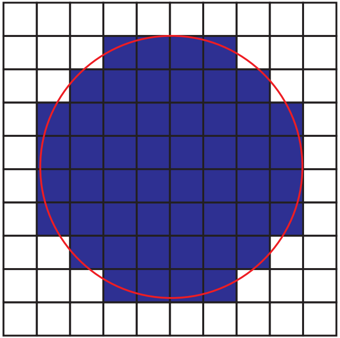
$d = 1$ grid unit | $d = 5$ grid units | $d = 10$ grid units | $d = 17$ grid units



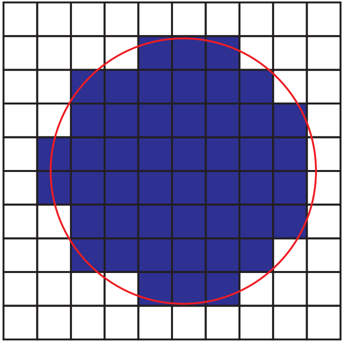
8×8 grid squares | 16×16 grid squares | 32×32 grid squares | 512×512 grid squares

Gauss Digitization: Properties

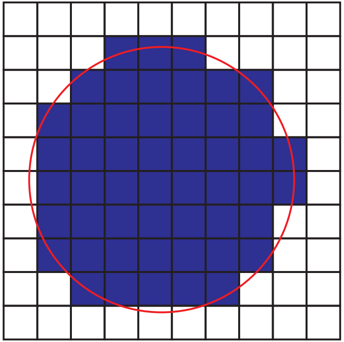
- The Gauss digitization $G_h(S)$ of any nonempty bounded set $S \subset \mathbb{R}^2$ is the union of a finite number of simple isothetic polygons
- Different sets can have identical Gauss digitizations
- The same sets after a rigid transformation can have different Gauss digitizations



An original disk



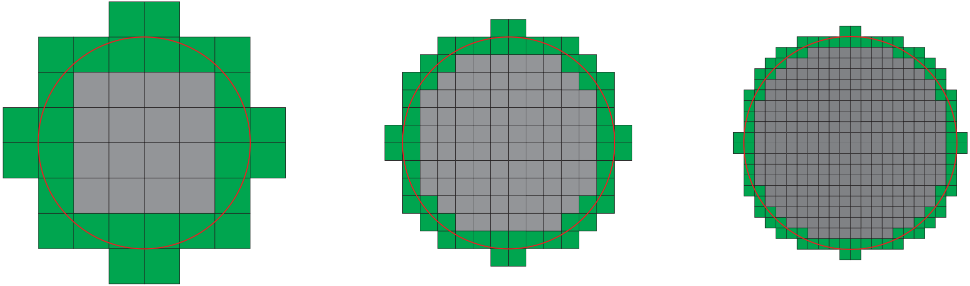
The disk shifted to the right



The disk shifted to the left

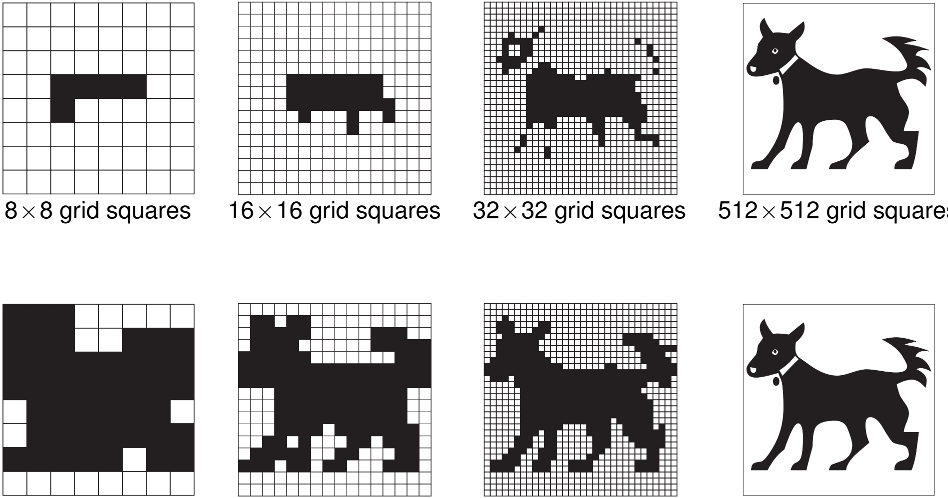
Jordan Digitization

- Originally defined for 3D grids only as Jordan used such grids to estimate the volumes of subsets of $\mathbb{R}^3$
- The inner Jordan digitization $J_h^-(S)$ of a planar set S on a 2D grid of resolution h is the union of the grid squares (2-cells) that are completely contained in S
- The outer Jordan digitization $J_h^+(S)$ of a planar set S on a 2D grid of resolution h is the union of the grid squares (2-cells) that have nonempty intersection with S



Inner (gray 2-cells) and outer (gray and green 2-cells) Jordan digitizations of a centered disk

Jordan Digitization: Examples

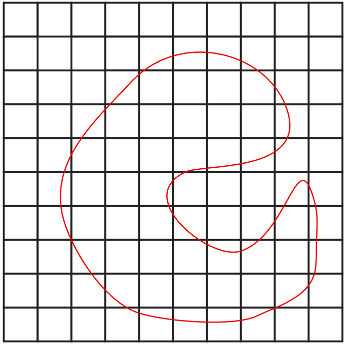


Inner: 8 × 8 grid squares, 16 × 16 grid squares, 32 × 32 grid squares, 512 × 512 grid squares

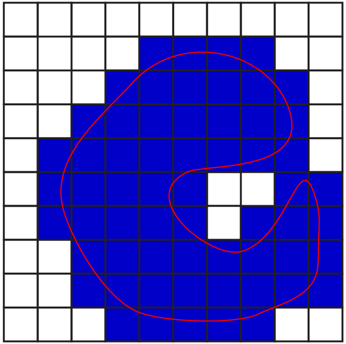
Outer: 8 × 8 grid squares, 16 × 16 grid squares, 32 × 32 grid squares, 512 × 512 grid squares

Jordan Digitization: Properties

- The inner and outer Jordan digitizations $J_h^-(S)$ and $J_h^+(S)$ of any nonempty bounded set $S \subset \mathbb{R}^2$ are the unions of finite numbers of simple isothetic polygons
- The outer Jordan digitization $J_h^+(S)$ of a connected set S is always a single connected isothetic polygon or polyhedron. However, it does not preserve simple connectedness because it can create holes



A real roll-like object S



The outer Jordan digitization of S

Relationships between Gauss and Jordan Digitizations

- Both digitization models are broadly used to digitize 2D and 3D sets

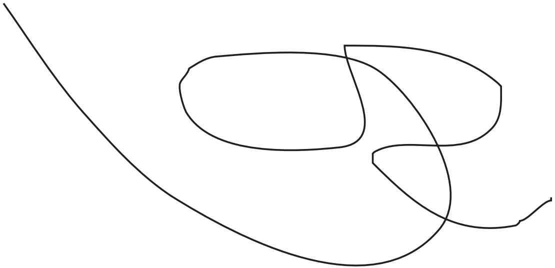
- They produce the same digitizations for:
  - Empty set: $J_h^-(\emptyset) = G_h(\emptyset) = J_h^+(\emptyset) = \emptyset$
  - Euclidean n-space $\mathbb{R}^n$ ($n \in \{2,3\}$): $J_h^-(\mathbb{R}^n) = G_h(\mathbb{R}^n) = J_h^+(\mathbb{R}^n) = \mathbb{R}^n$
  - Finite unions of n-cells in nD ($n \in \{2,3\}$)

- The obtained digitizations are ordered by inclusion:

$$J_h^-(S) \subseteq G_h(S) \subseteq J_h^+(S) \text{ for any } S \subseteq \mathbb{R}^2 \ (S \subseteq \mathbb{R}^3)$$

Grid-Intersection Digitization

- Neither Gauss nor inner Jordan digitization is appropriate for the digitization of 1D sets (curves). Outer Jordan digitization is appropriate but grid-intersection digitization is the preferred choice for curves
- The grid-intersection digitization $R(\gamma)$ of a planar curve γ is the set of all grid points with closest Euclidean distances to the intersection points of γ with the grid lines
- In case an intersection point is of the same distance from two grid points, either both grid points are added to $R(\gamma)$ or one of them is chosen based on a predefined rule



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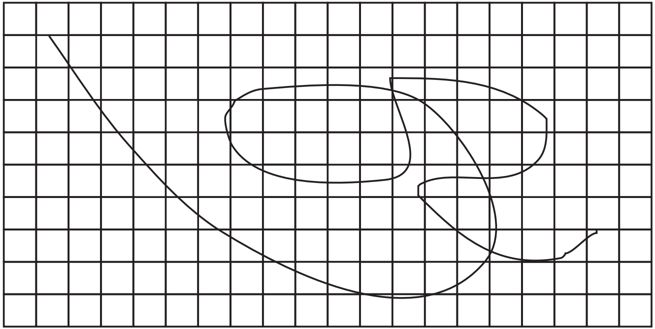

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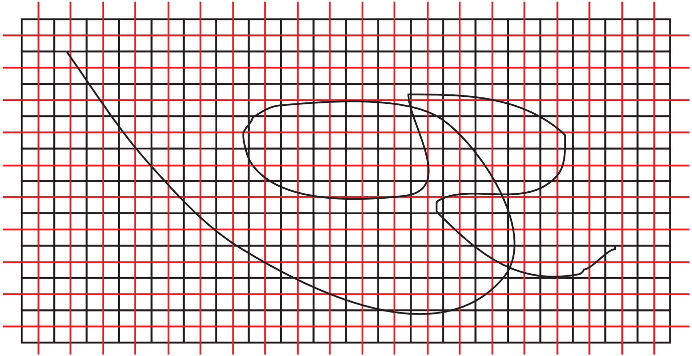

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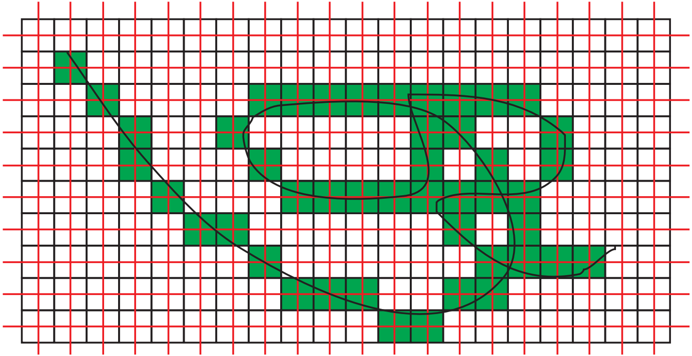

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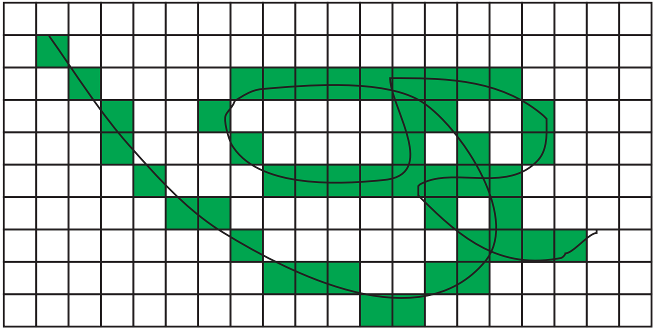
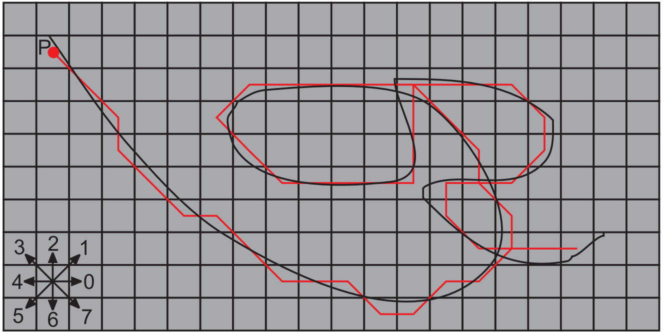


Digitized Grid=Intersection Sequence

- An ordered sequence of grid points in $R(\gamma)$ is called a digitized grid-intersection sequence $\rho(\gamma)$ of γ
- Such a sequence can be represented by a chain code
- Remark: Chain codes can also represent object borders (typically obtained by a border tracing algorithm)



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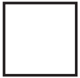
DOMAIN DIGITIZATIONS

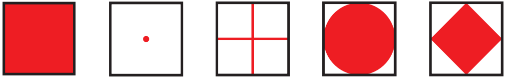
Preliminaries

- We want to define a framework for a general class of digitization models in nD

- Let $\Pi_{cube} = \left\{ (x_1, \ldots, x_n) : \max_{1 \le i \le n} |x_i| \le \frac{1}{2} \right\}$ be a n-cell centered at $o = (0, \ldots, 0)$:



- Let $\emptyset \neq \Pi_\sigma \subseteq \Pi_{cube}$, and consider its translates $\Pi_\sigma(q) = \{q + p : p \in \Pi_\sigma\}$ centered at grid points $q \in \mathbb{Z}^n$ as the domains of influence:



- Obviously, $\Pi_{cube}(q)$ is the n-cell c_q centered at q

σ-Digitization

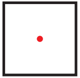
- The inner σ-digitization $dig_\sigma^-(S)$ of a set $S \subseteq \mathbb{R}^n$ is the union of all c_q such that $\Pi_\sigma(q)$ is contained in S:

$$c_q \subseteq dig_\sigma^-(S) \text{ iff } \Pi_\sigma(q) \subseteq S$$

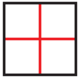
- The outer σ-digitization $dig_\sigma^+(S)$ of a set $S \subseteq \mathbb{R}^n$ is the union of all c_q such that $\Pi_\sigma(q)$ intersects S:

$$c_q \subseteq dig_\sigma^+(S) \text{ iff } \Pi_\sigma(q) \cap S \neq \emptyset$$

If $\Pi_\sigma = \Pi_{cube}$, $dig_{cube}^- = J^-$ (inner Jordan digitization) and $dig_{cube}^+ = J^+$ (outer Jordan digitization)



If $\Pi_\sigma = \{o\}$, $dig_\sigma^+ = dig_\sigma^- = G$ (Gauss digitization)



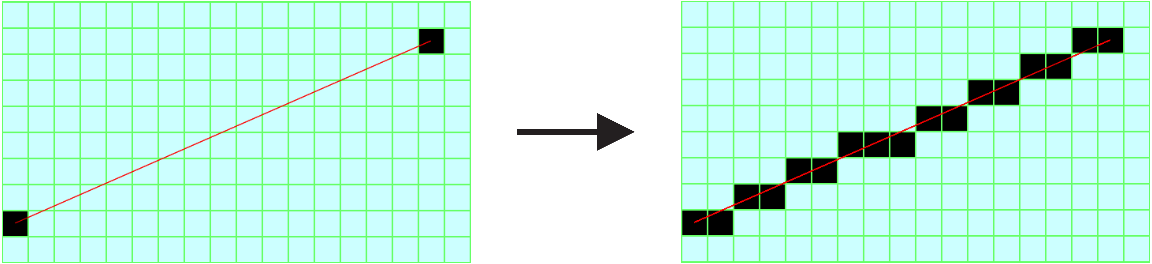
If $\Pi_\sigma = \{(x_1, \ldots, x_n) : \exists i.(1 \leq i \leq n \wedge x_i = 0) \wedge \max_{1 \leq i \leq n} |x_i| \leq \frac{1}{2})\}$,

$dig_\sigma^+ = R$ (grid-intersection digitization)

DIGITIZATION OF STRAIGHT LINES

Bresenham's Algorithm for Line Digitization

- A standard routine in computer graphics, which builds on top of the grid-intersection digitization model



Check out a demo at `http://bert.stuy.edu/pbrooks/graphics/demos/BresenhamDemo.htm`

Bresenham's Algorithm (First Octant, Nonnegative Slope)

Task: Draw a digital line with a nonnegative slope between two points, (x_s, y_s) and (x_e, y_e), in the first octant

Pseudocode of the algorithm

1. Initialize constants: $\texttt{dx} = x_e - x_s$, $\texttt{dy} = y_e - y_s$, $\texttt{b0} = 2 * \texttt{dy}$, $\texttt{b1} = 2 * (\texttt{dy} - \texttt{dx})$
2. Initialize variables: $\texttt{x} = x_s$, $\texttt{y} = y_s$, $\texttt{err} = 2 * \texttt{dy} - \texttt{dx}$
3. $\texttt{while}\ \texttt{x} \leq x_e$

   Draw $(\texttt{x}, \texttt{y})$ as a digital line element

   $\texttt{x} = \texttt{x} + 1$

   $\texttt{if}\ \texttt{err} < 0$

   $\quad \texttt{err} = \texttt{err} + \texttt{b0}$

   $\texttt{else}$

   $\quad \texttt{y} = \texttt{y} + 1$

   $\quad \texttt{err} = \texttt{err} + \texttt{b1}$

Complexity: The algorithm runs in $\mathcal{O}(x_e - x_s)$ and involves basic assignment, arithmetic, and conditional operations only

Take-Home Messages

- Digital geometric figures (shapes) are sets of grid points obtained by digitizing their continuous counterparts
- 1D sets (curves) are digitized using the grid-intersection digitization model
- 2D and 3D sets are digitized using the Gauss or Jordan digitization models
- Domain digitization defines a general digitization model
- The Bresenham algorithm digitizes lines using the grid-intersection digitization model