PA170 Digital Geometry

Lecture 03: Metrics

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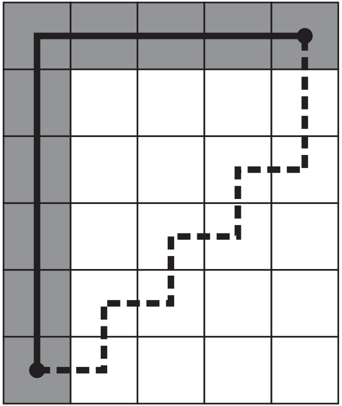
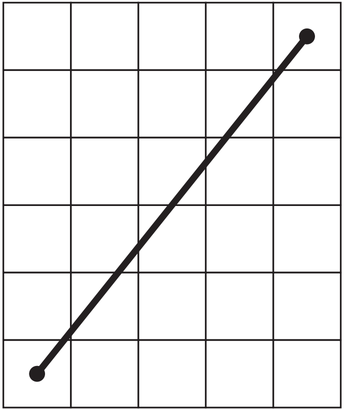
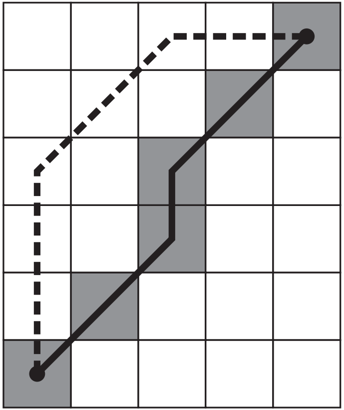
Motivation: How To Measure Distances in Digital Grids



City-block distance



Euclidean distance



Chessboard distance

INTRODUCTION TO METRICS

- Let $[S, +, \cdot, \mathbb{R}]$ be an n-dimensional vector space over $\mathbb{R}$. A norm $\|\cdot\|$ assigns to any $p \in S$ a nonnegative scalar $\|p\|$ that satisfies the following three properties:

N1 Identity
$\forall p \in S : \|p\| = 0 \text{ iff } p = (0, \ldots, 0)$

N2 Homogeneity
$\forall p \in S, \forall a \in \mathbb{R} : \|a \cdot p\| = |a| \cdot \|p\|$

N3 Triangle Inequality
$\forall p, q \in S : \|p + q\| \leq \|p\| + \|q\|$

Metrics

- Let S be an arbitrary nonempty set. A function $d : S \times S \to \mathbb{R}$ is a distance function (metric) on S iff it has the following three properties:

  M1 Positive Definiteness
  $\forall p, q \in S : d(p, q) \geq 0$ and $d(p, q) = 0$ iff $p = q$

  M2 Symmetry
  $\forall p, q \in S : d(p, q) = d(q, p)$

  M3 Triangle Inequality
  $\forall p, q, r \in S : d(p, r) \leq d(p, q) + d(q, r)$

- If $\| \cdot \|$ is a norm on $[S, +, \cdot, \mathbb{R}]$, $d(p, q) = \|p - q\|$ $(\forall p, q \in S)$ defines a norm-induced metric on S

- A norm-induced metric has also the following two properties:

  M4 Translation Invariance
  $\forall p, q, r \in S : d(p + r, q + r) = d(p, q)$

  M5 Homogeneity
  $\forall p, q \in S, \forall a \in \mathbb{R} : d(a \cdot p, a \cdot q) = |a| \cdot d(p, q)$

Restricting and Combining Metrics

- If $[S, d]$ is a metric space and $\emptyset \neq A \subseteq S$, $[A, d]$ is also a metric space
- Metrics on $\mathbb{R}^n$ define metrics on $\mathbb{Z}^n$, $\mathbb{G}_{m,n}$, and $\mathbb{G}_{l,m,n}$

- If d is not a metric on A, d is not a metric on any set S containing A either

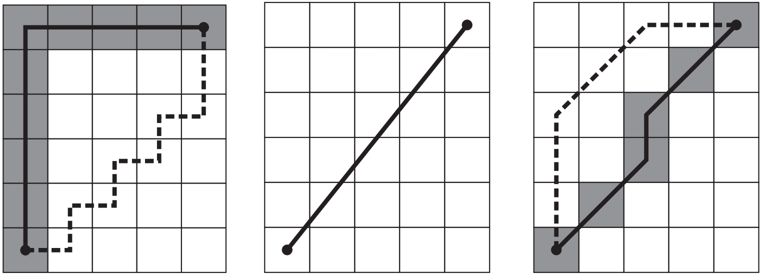
- Any positive linear combination $a \cdot d_1 + b \cdot d_2$ and maximum $\max\{d_1, d_2\}$ of two metrics d_1 and d_2 on a set S define a metric on S
- The product $d_1 \cdot d_2$ and minimum $\min\{d_1, d_2\}$ are not necessarily metrics on S

Minkowski Norms and Metrics

- Let $S = \mathbb{R}^n$ and $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$
- The Minkowski norms $\|p\|_m$ on $[S, +, \cdot, \mathbb{R}]$ are defined as:
  - $\|p\|_m = \sqrt[m]{|x_1|^m + \cdots + |x_n|^m} \qquad (m = 1, 2, \ldots)$
  - $\|p\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$
- The Minkowski norms $\|p\|_m$ on $[S, +, \cdot, \mathbb{R}]$ induce the Minkowski metrics L_m on S:
  - $L_m(p, q) = \sqrt[m]{|x_1 - y_1|^m + \cdots + |x_n - y_n|^m} \qquad (m = 1, 2, \ldots)$
  - $L_\infty(p, q) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$
- A sequence of Minkowski distances L_m for increasing m is nondecreasing:

$$\forall p, q \in \mathbb{R}^n, 1 \leq m_1 \leq m_2 \leq \infty : L_{m_1}(p, q) \geq L_{m_2}(p, q)$$

- Let $p = (x_p, y_p) \in \mathbb{Z}^2$ and $q = (x_q, y_q) \in \mathbb{Z}^2$, we define
  - City-block metric $d_4(p, q) = |x_p - x_q| + |y_p - y_q| = L_1(p, q)$
  - Euclidean metric $d_e(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} = L_2(p, q)$
  - Chessboard metric $d_8(p, q) = \max\{|x_p - x_q|, |y_p - y_q|\} = L_\infty(p, q)$

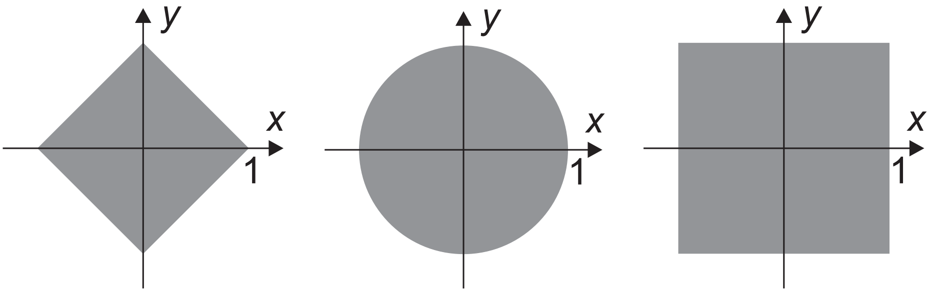


City-block metric ($d_4 = L_1$) — Euclidean metric ($d_e = L_2$) — Chessboard metric ($d_8 = L_\infty$)

- $d_8(p, q) \leq d_e(p, q) \leq d_4(p, q) \leq 2 \cdot d_8(p, q) \quad (\forall p, q \in \mathbb{Z}^2)$

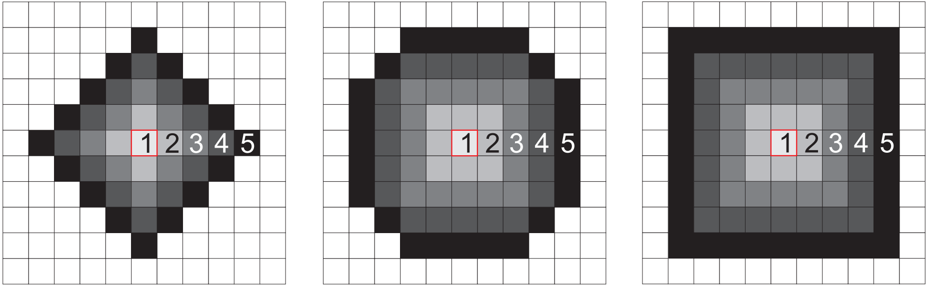
Unit Disks

- Let $S \subseteq \mathbb{R}^2$, $o = (0,0) \in S$, and d be a metric on S. The set $\{p \in S : d(p,o) \leq 1\}$ is called a unit disk in $[S,d]$
- Translation-invariant (**M4**) and homogeneous (**M5**) metrics can be compared via their unit disks



City-block metric ($d_4 = L_1$) | Euclidean metric ($d_e = L_2$) | Chessboard metric ($d_8 = L_\infty$)

- Let $0 < e \in \mathbb{R}$ and d be a metric on a grid $\mathbb{G}$. The set $N_{e,d}(p) = \{q \in \mathbb{G} : d(p,q) < e\}$ is called an *e*-neighborhood of a grid point $p \in \mathbb{G}$ for the metric d



$N_{e,d_4}(p)$ $N_{e,d_e}(p)$ $N_{e,d_8}(p)$

Integer-Valued Metrics

- In digital geometry, the measurements are often based on integer-valued metrics
- In contrast to d_4 and d_8, d_e is not an integer-valued metric on digital grids

- Let $a \in \mathbb{R}$, we define
  - $\lfloor a \rfloor$ is the largest integer less than or equal to a (`floor`)
  - $\lceil a \rceil$ is the smallest integer greater than or equal to a (`ceil`)
  - $[a]$ is the nearest integer to a if it is unique, and $\lceil a \rceil$ otherwise (`round`)

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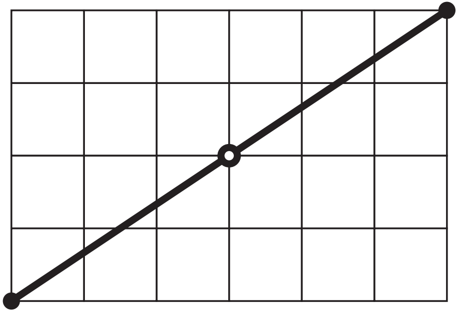
$\lfloor d_e \rfloor$ and $[d_e]$ are not metrics on $\mathbb{Z}^n$ (**M3** is broken)
$\lceil d_e \rceil$ is a metric on $\mathbb{Z}^n$ (If d is a metric on S, $\lceil d \rceil$ is also a metric on S)

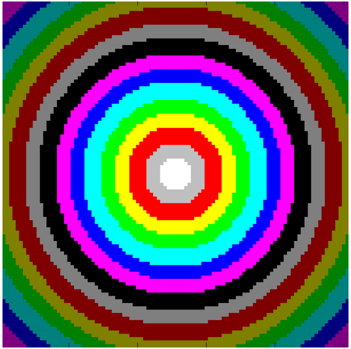
Regular Metrics

- An integer-valued metric d on a set S is called regular iff, for all $p, q \in S$ such that $d(p, q) \geq 2$, there exists $r \in S$ ($r \neq p$ and $r \neq q$) such that $d(p, q) = d(p, r) + d(r, q)$
- It implies that for all distinct $p, q \in S$, there exists $r \in S$ such that $d(p, r) = 1$ and $d(p, q) = 1 + d(r, q)$
- d_4 and d_8 are regular integer-valued metrics on $\mathbb{Z}^2$
- $\lceil d_e \rceil$ is a regular integer-valued metric on $\mathbb{R}^n$ but not on $\mathbb{Z}^n$ ($n > 1$)

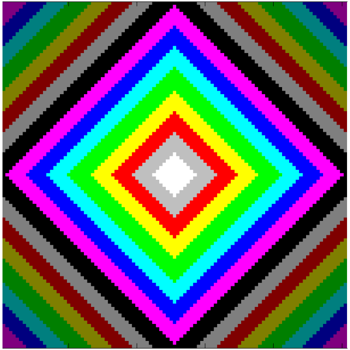


APPROXIMATION TO THE EUCLIDEAN METRIC

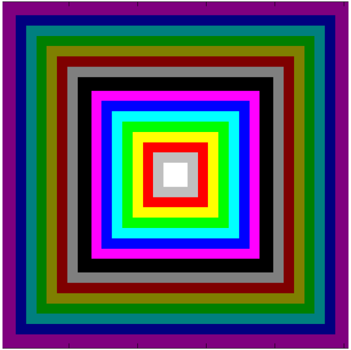
Motivation: d_4 and d_8 Are Too Coarse Approximations to d_e



e-Neighborhoods of d_e



e-Neighborhoods of d_4

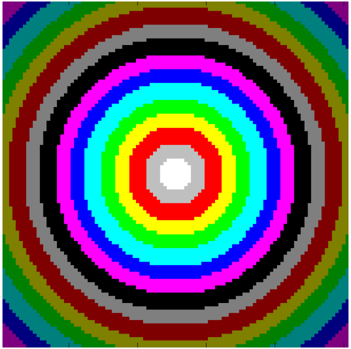


e-Neighborhoods of d_8

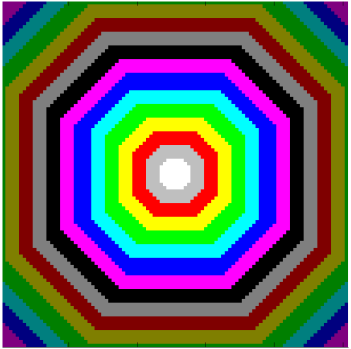
- We can combine d_4 and d_8 as

$$d(p,q) = \max\{d_8, \frac{2}{3} \cdot d_4\}$$

- e-Neighborhoods of $d(p,q)$ are upright octagons obtained by intersecting upright squares of side $2 \cdot e$ with diamonds of diagonal $3 \cdot e$
- Regular octagons can be reached by choosing the pair of weights appropriately



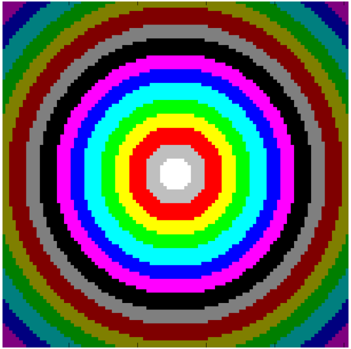
e-Neighborhoods of d_e



e-Neighborhoods of $\max\{d_8, \frac{2}{3} \cdot d_4\}$

Chamfer Distance

- Let $p, q \in \mathbb{Z}^2$, and let ρ be a sequence of king's moves from p to q
- Let $l_{a,b}(\rho) = a \cdot m + b \cdot n$ with m being the number of isothetic moves and n being the number of diagonal moves
- $d_{a,b}(p,q) = \min_{\rho} l_{a,b}(\rho)$ is called the (a,b)-chamfer distance from p to q
- Generalized chamfer distances can be defined using additional types of moves



e-Neighborhoods of d_e

e-Neighborhoods of $d_{1,\sqrt{2}}$

Chamfer Distance: Properties

- The chamfer distance $d_{a,b}$ is a metric if $0 < a \leq b \leq 2a$
- This metric is a nonnegative linear combination of d_4 and d_8

- On a $(k+1)\times(k+1)$ grid, the chamfer distance $d_{1,b}$ that best approximates d_e has

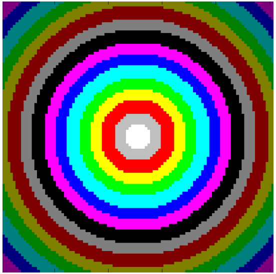
$$b = \frac{1}{\sqrt{2}} + \sqrt{\sqrt{2}-1} \approx 1.351 \, ,$$

producing a maximum error of

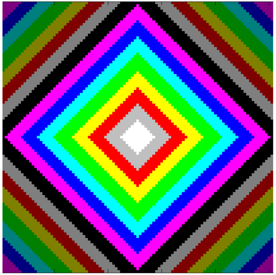
$$|d_e - d_{1,b}| \leq \left(\frac{1}{\sqrt{2}} - \sqrt{\sqrt{2}-1} \right) k \approx 0.06k$$

- The optimal value of b is close to $\frac{4}{3}$, and thus $d_{3,4}$ is a good approximation to $3 \cdot d_e$

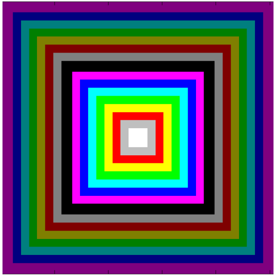
Summary: Different Approximations to the Euclidean metric



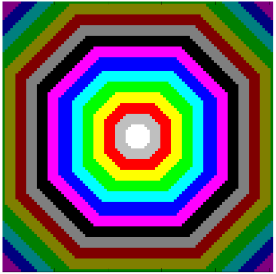
d_e



d_4



d_8



$\max\{d_8, \frac{2}{3} \cdot d_4\}$

$d_{1,\sqrt{2}}$ (chamfer distance)

Take-Home Messages

- The city-block (d_4) and chessboard (d_8) metrics are regular, integer-valued metrics on digital grids
- $\lceil d_e \rceil$ is an integer-valued metric on digital grids, but it is not regular
- The chamfer distance $d_{3,4}$ is a regular, integer-valued metric on 2D digital grids, which provides a good approximation to $3 \cdot d_e$