PA170 Digital Geometry

Lecture 05: Adjacency Graphs

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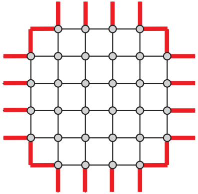
Boundaries, Borders, and Holes Informally

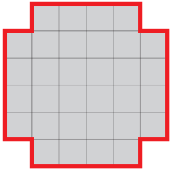
- What is a boundary, border, and hole of a set?

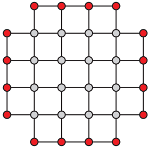
Boundaries, Borders, and Holes Informally

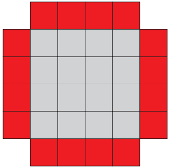
- What is a boundary, border, and hole of a set?

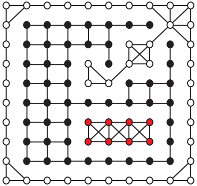
- Boundary lies "between" two sets and separates them from each other
- Border is a part of a set that lies near its boundary
- Hole is a component "inside" a set (a finite complementary component)


Boundary


Boundary

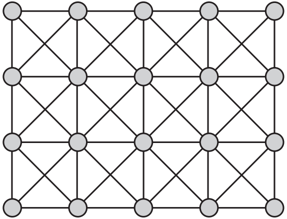

Border


Border

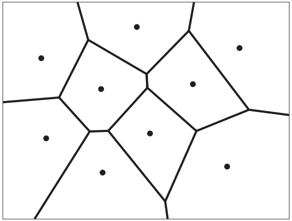

Hole

Adjacency Structures as Adjacency Graphs

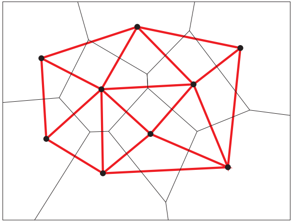
- Let S be a countable set and A be an adjacency relation on S. $[S, A]$ is called an adjacency structure
- An adjacency structure $[S, A]$ can be represented using an adjacency graph (its nodes correspond to the elements of S and its edges are given by A)
- Nodes of adjacency graphs do not have assigned locations in a Euclidean space



8-adjacency grid



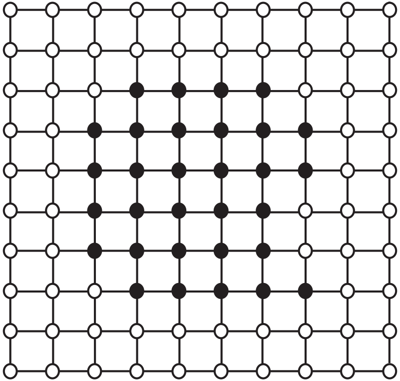
Voronoi tesselation



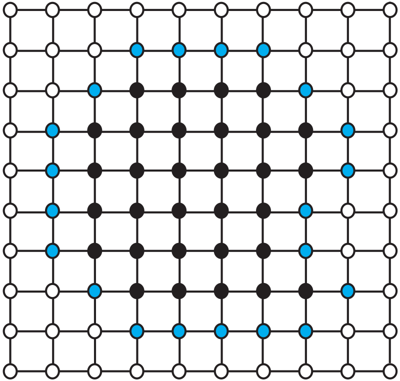
Delaunay triangulation

Adjacency Sets

- Let $[S, A]$ be an adjacency graph. The set $A(M)$ of all nodes adjacent to $M \subseteq S$ (i.e., the set of all $p \in \overline{M}$ such that $A(p) \cap M \neq \emptyset$) is called the adjacency set of M



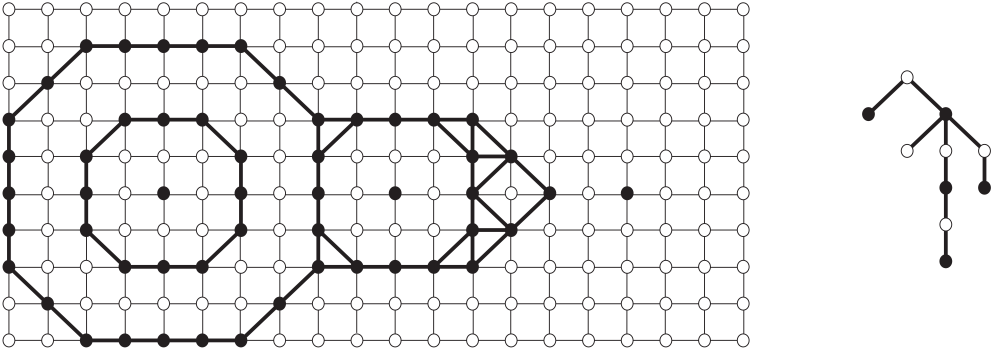
Set M in a 4-adjacency graph



4-adjacency set of M

Region Adjacency Graphs

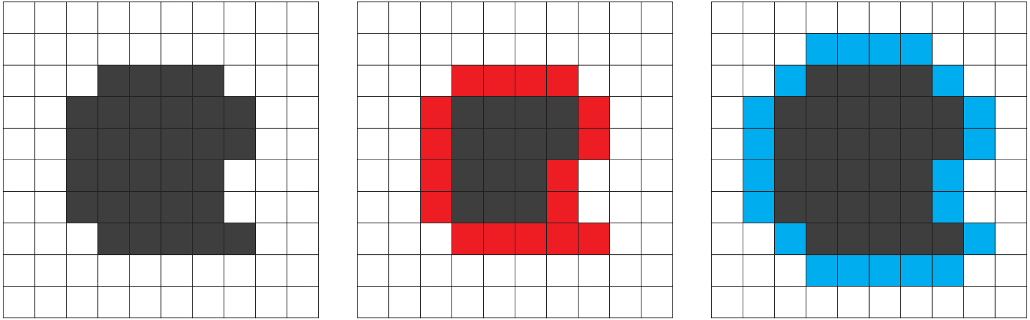
- If $[S, A]$ is an adjacency graph, two disjoint subsets M_1 and M_2 of S are called adjacent ($M_1 \mathcal{A} M_2$ or $(M_1, M_2) \in \mathcal{A}$) iff $A(M_1) \cap M_2 \neq \emptyset$
- A region adjacency relation $\mathcal{A}$ is irreflexive and symmetric
- Let $\mathcal{R}$ be a partition of S into connected components and (possibly) the infinite background component. The undirected graph $[\mathcal{R}, \mathcal{A}]$ is called the region adjacency graph of $\mathcal{R}$



$(8, 4)$-adjacency graph and the corresponding region adjacency graph

Types of Nodes and Borders

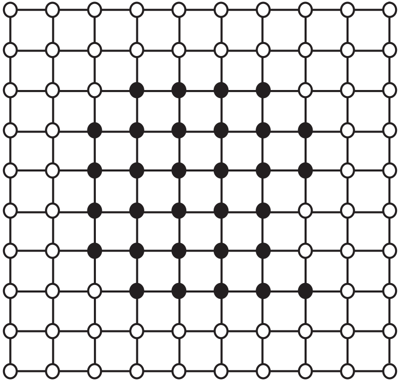
- Let $[S, A]$ be an adjacency graph and $M \subseteq S$ be a subset of its nodes
- $p \in M$ is called an interior node of M iff $A(p) \subseteq M$; otherwise it is called a border node of M
- The set M^{∇} of all inner nodes of M is called the inner set of M
- The set δM of all border nodes of M is called the border (inner border) of M
- A border node of $S \setminus M$ is sometimes called a coborder (outer border) node of M



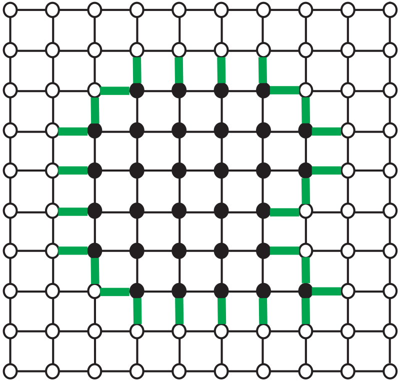
Set M — Its inner 4-border — Its outer 4-border

Invalid Edges and Boundary

- Let $[S, A]$ be an adjacency graph and $M \subset S$ be a proper subset of S
- An edge $\{p, q\}$ of $[S, A]$ is called M-invalid iff $p \in M$ and $q \in (S \setminus M)$
- The boundary of M is the set of all M-invalid edges



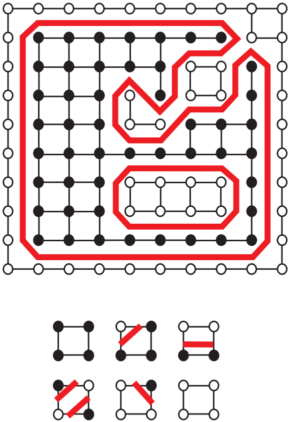
Set M in a 4-adjacency graph



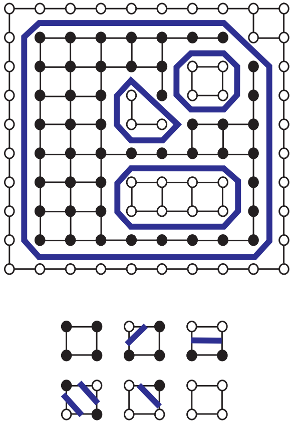
Boundary of M

Boundary Detection

- Detection of boundaries in 2D (3D) binary images is often performed using the Marching Squares (Cubes) algorithm
- It scans the input binary image in successive blocks of image elements and evaluates their configurations of foreground and background values assigned



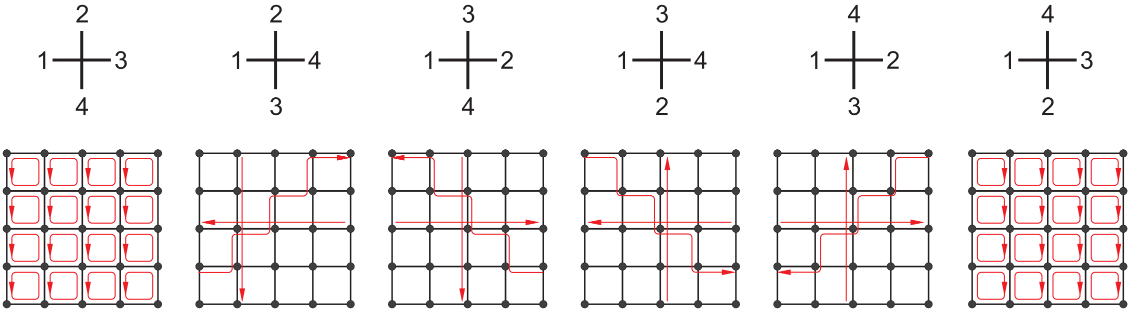
Foreground boundary for $(4, 8)$-adjacency



Foreground boundary for $(8, 4)$-adjacency

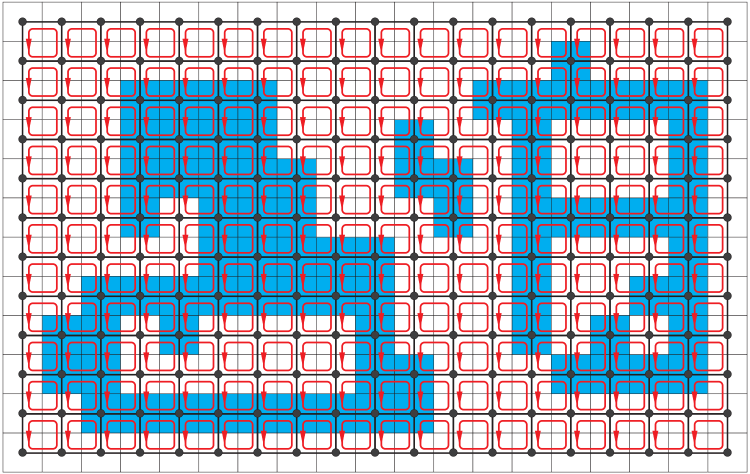
Oriented Adjacency Graphs

- Let $[S, A]$ be an adjacency graph. A local circular order $\xi(p)$ at a node $p \in S$ is an ordered sequence $\langle q_1, \ldots, q_n \rangle$ of all nodes in $A(p)$ without repetitions
- An oriented adjacency graph $[S, A, \xi]$ is defined by an adjacency graph $[S, A]$ and an orientation ξ, defined by local circular orders of the adjacency sets
- These local orders can be used to trace edges in $[S, A, \xi]$ as follows: if we arrive at p from $q_i \in A(p)$, we move next to q_k, where $k = (i + 1) \bmod n$



Oriented Adjacency Subgraphs

- A subset $M \subseteq S$ induces a substructure $[M, A_M, \xi_M]$ of an oriented adjacency graph $[S, A, \xi]$ where A_M contains only those adjacency pairs $\{p, q\}$ such that $p, q \in M$ and $\{p, q\} \in A$, and where, for any $p \in M$, $\xi_M(p)$ is the reduced local circular order defined by deleting all nodes that are not in M from $\xi(p)$
- The cycles of $[M, A_M, \xi_M]$ may differ from cycles of $[S, A, \xi]$



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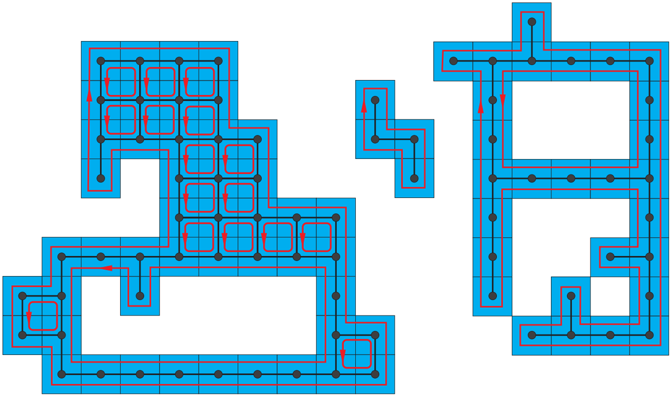
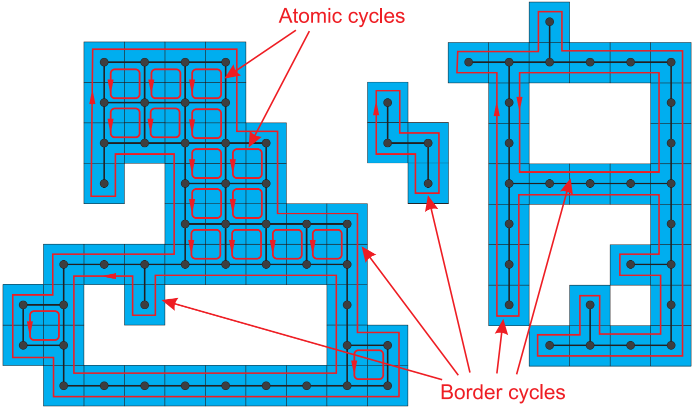


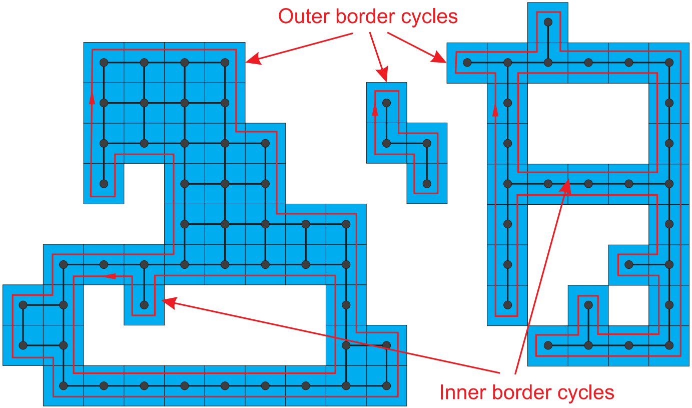
Atomic and Border Cycles

- Let (p,q) be a directed edge in $[M, A_M, \xi_M]$, ρ_1 be the cycle generated by (p,q) in $[M, A_M, \xi_M]$, and ρ_2 be the cycle generated by (p,q) in $[M, A, \xi]$. ρ_1 is an atomic cycle of $[M, A_M, \xi_M]$ iff $\rho_1 = \rho_2$ and a border cycle of $[M, A_M, \xi_M]$ otherwise
- For any $M \subset S$, $[M, A_M, \xi_M]$ has at least one border cycle
- Each border cycle of $[M, A_M, \xi_M]$ contains at least one border node of M, and each border node is incident with at least one border cycle



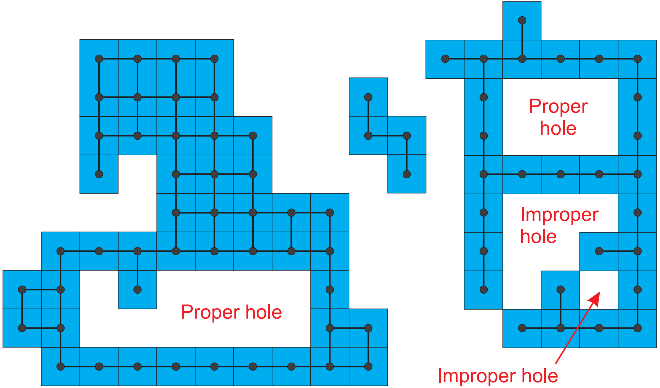
Inner and Outer Border Cycles

- A border cycle of $[M, A_M, \xi_M]$ is called outer iff it separates M from the infinite complementary component
- All other border cycles of $[M, A_M, \xi_M]$ are called inner border cycles



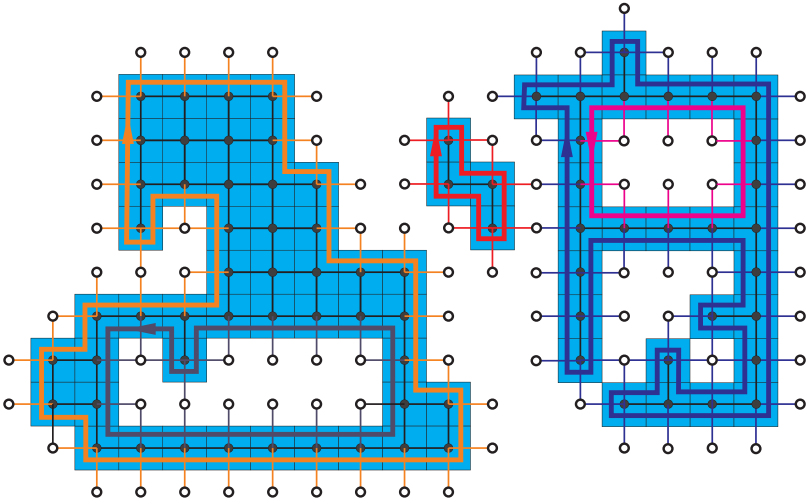
Holes

- Let $[S, A, \xi]$ be an infinite oriented adjacency graph and M be a finite connected subset of S with exactly one infinite complementary component
- Any finite complementary component of M is called a hole of M
- A (finite) complementary component of M assigned to an inner border cycle is called a proper hole of M
- A finite complementary component assigned to an outer border cycle of M is called an improper hole of M



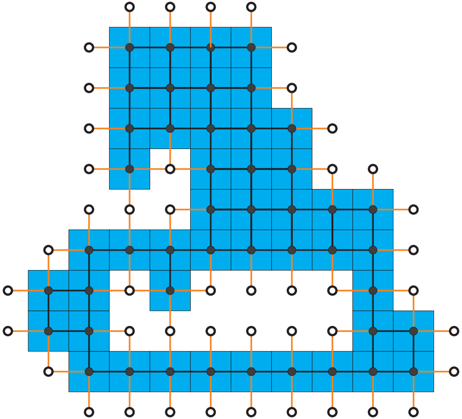
Directed Invalid Edges

- Let $[S, A, \xi]$ be an oriented adjacency graph and $M \subseteq S$ be a subset of its nodes. A directed edge (r, q) from a coborder node $r \in (S \setminus M)$ to a boder node $q \in \delta M$ is called invalid
- Every directed invalid edge points to exactly one border cycle in $[M, A_M, \xi_M]$



Border Tracing Algorithm: Main Idea

- Let $[S, A, \xi]$ be an oriented adjacency graph and $M \subset S$ be a finite proper subset of S
- Each component C of M is handled independently:
  - Generate a list of all directed invalid edges for C
  - Trace the border cycle to which one of these edges points to (and delete the already processed invalid edges)
  - If there is still an undeleted directed invalid edge, repeat the tracing process



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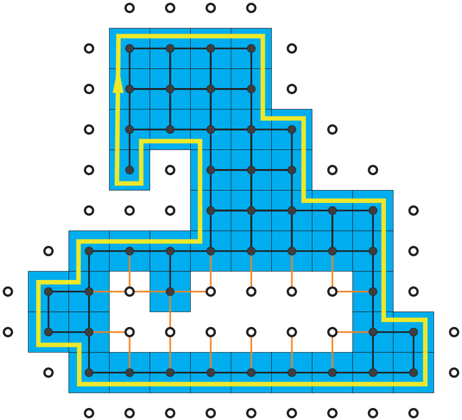

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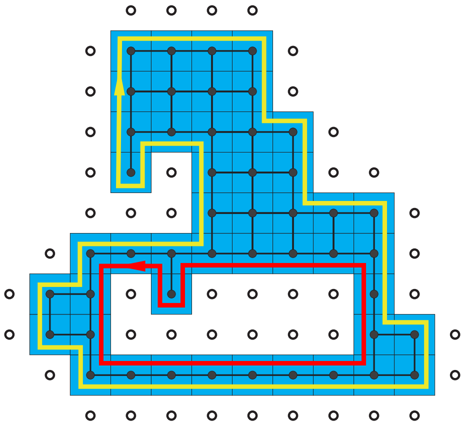

Border Tracing Algorithm: Directed Invalid Edge (q, p)

1. Let $(q_0, p_0) := (q, p)$, $i := 0$, and $k := 0$

2. Let $\xi(p_i) = \langle \dots, q_k, q, \dots \rangle$ be the local circular order at p_i. If $q \in \overline{C}$, go to Step 4

3. Node q is another node on the border cycle. Let $i := i + 1$ and $p_i := q$. Let $\xi(p_i) = \langle \dots, p_{i-1}, q, \dots \rangle$ be the local circular order at p_i. If $q \in C$, go to Step 3; Otherwise, let $k := i - 1$, and go to Step 4

4. If $(q, p_i) = (q_0, p_0)$, go to Step 5. Otherwise, let $k := k + 1$ and $q_k := q$, and go to Step 2

5. We are back at the original directed invalid edge (q, p). The border cycle of the component C is $\langle p_0, p_1, \dots, p_i \rangle$

Boundaries, Borders, and Holes Formally

- What is a boundary, border, and hole of a set M?

Boundaries, Borders, and Holes Formally

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- The boundary of M is the set of all M-invalid edges of the corresponding adjacency graph
- The border δM of M is the set of all border nodes of the corresponding adjacency graph
- Proper and improper holes of M are finite complementary components of M assigned to the border cycles of the corresponding oriented adjacency graph