PA170 Digital Geometry

Lecture 06: Introduction to Topology

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Topology in a Nutshell

- Topology is a branch of mathematics, which studies properties of spaces under continuous deformations
- It is often viewed as "rubber-sheet geometry" because objects can be stretched and contracted like rubber, but they cannot be broken nor glued together
- Joke: A topologist is a person who cannot distinguish a coffee mug from a doughnut

A coffee mug　　　　A doughnut

Source: https://en.wikipedia.org/wiki/Topology　https://jcdverha.home.xs4all.nl/scijokes

Common Subfields of Topology

Point Set Topology

- It considers local properties of spaces, and is closely related to analysis
- It generalizes the concept of continuity to define topological spaces, in which limits of sequences can be considered

Combinatorial Topology

- It is the oldest branch of topology, which dates back to Euler
- It considers the global properties of spaces, built up from a network of vertices, edges, and faces

Algebraic Topology

- It also considers the global properties of spaces, and uses algebraic objects such as groups or rings to answer topological questions
- It converts topological problems into algebraic ones that are hopefully easier to solve

Differential Topology

- It considers spaces with some kind of smoothness associated to each point (e.g., a square and a circle are not differentiably equivalent to each other)
- It is useful for studying properties of vector fields, such as magnetic or electric fields

TOPOLOGICAL SPACES

Topological Spaces: Basic Terms

- $[S, \mathcal{G}]$ is called a topological space iff $\mathcal{G}$ is a family of subsets of S with the following three properties:
  - **T1:** $\{\emptyset, S\} \subseteq \mathcal{G}$
  - **T2:** Let $M_1, M_2, \ldots$ be a finite or infinite family of sets in $\mathcal{G}$. The union of these sets is also in $\mathcal{G}$
  - **T3:** Let $M_1, M_2, \ldots, M_n$ be a finite family of sets in $\mathcal{G}$. The intersection of these sets is also in $\mathcal{G}$
- $\mathcal{G}$ is called a topology on S, and its elements are called open sets
- A set $M \subseteq S$ is called closed iff its complement $\overline{M} = S \setminus M$ is open
- M is called (topologically) connected iff it is not the union of two disjoint non-empty open subsets of M
- The interior M° of $M \subseteq S$ is the union of all open subsets of M
- The closure $M^\bullet$ of M is the intersection of all closed subsets of S that contain M
- The frontier of M is the set $\partial M = M^\bullet \cap (\overline{M})^\bullet$

Homeomorphism and Topological Invariants

- Let Φ be a mapping of a topological space S_1 into a topological space S_2
- Φ is called continuous iff, for any open subset M of S_2, the set $\Phi^{-1}(M) = \{p \in S_1 : \Phi(p) \in M\}$ is open in S_1
- Φ is called a homeomorphism iff it is bijective, continuous, and Φ^{-1} is also continuous

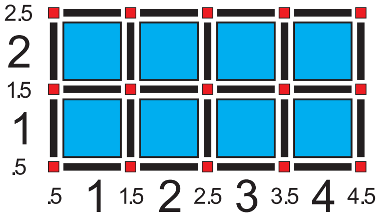
- Two topological spaces are called homeomorphic (topologically equivalent) iff each of them can be mapped onto the other by a homeomorphism
- The letters I and C are homeomorphic, whereas the letters X and Y are not

- A property of a subset M of a topological space S is called a topological invariant iff, for any homeomorphism Φ, the property is also valid for $\Phi(M)$
- The number of components and Euler characteristic are examples of topological invariants

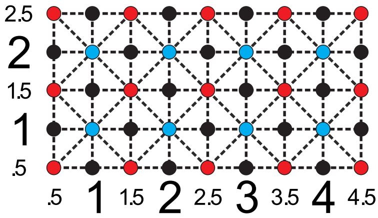
TOPOLOGY ON INCIDENCE GRAPHS

Incidence Grids as Incidence Graphs

- A regular incidence grid can be represented by an incidence graph $[S, I, \mathrm{dim}]$
- Remark: The incidence relation I is reflexive (self-incidence is allowed), and thus incidence graphs are pseudographs (they contain loops) formally

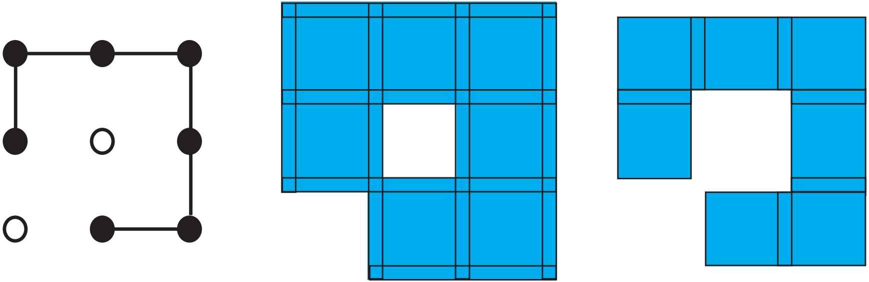


0-, 1-, and 2-cells



Incidence graph

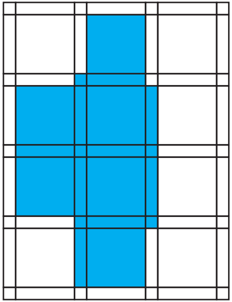
- Let $G = [S, I, \mathrm{dim}]$ be an incidence graph and $M \subseteq S$ be a subset of its nodes
- M is called closed in G iff, for any $p \in M$ and any $q \in I(p)$ such that $\mathrm{dim}(q) < \mathrm{dim}(p)$, we have $q \in M$
- M is called open in G iff $\overline{M} = S \setminus M$ is closed in G
- The closure $M^\bullet$ of M is the smallest closed set that contains M
- Remark: These definitions do not have analogues for adjacency graphs



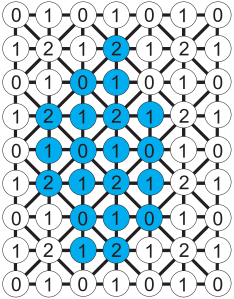
Set M (point grid) | Its closed representation (incidence grid) | Its open representation (incidence grid)

Inner and Border Nodes

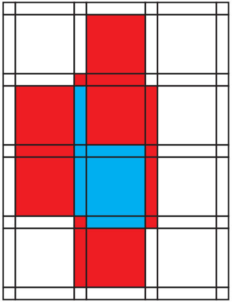
- Let $[S, I, \dim]$ be an incidence graph and $M \subseteq S$ be a subset of its nodes
- A node $p \in M$ is called an inner node of M iff $I(p) \subseteq M$. Otherwise, it is called a border node of M
- The set of all inner nodes of M is called the inner set M^{∇} of M, and the set of border nodes of M is called the border δM of M



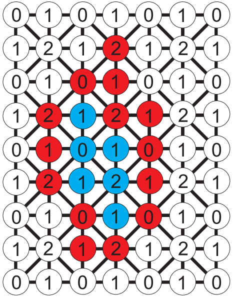
Set M
(incidence grid)



Set M
(incidence graph)

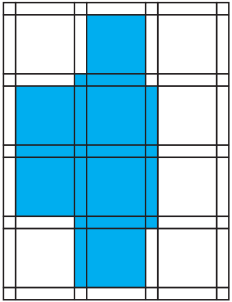


Its border
(incidence grid)

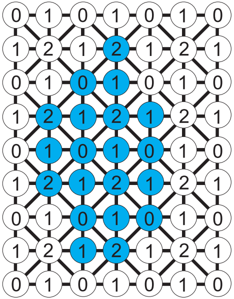


Its border
(incidence graph)

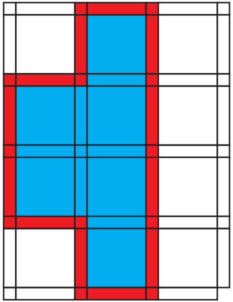
- Let $[S, I, \dim]$ be an incidence graph and $M \subseteq S$ be a subset of its nodes. The frontier ϑM of M is the border of $M^\bullet$
- Remark: This definition does not have an analogue for adjacency graphs



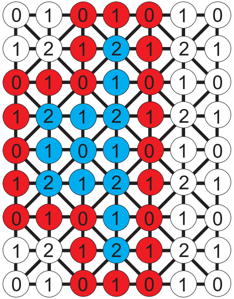
Set M
(incidence grid)



Set M
(incidence graph)



Its frontier
(incidence grid)



Its frontier
(incidence graph)

DIGITAL TOPOLOGY

Digital Topology: Definition

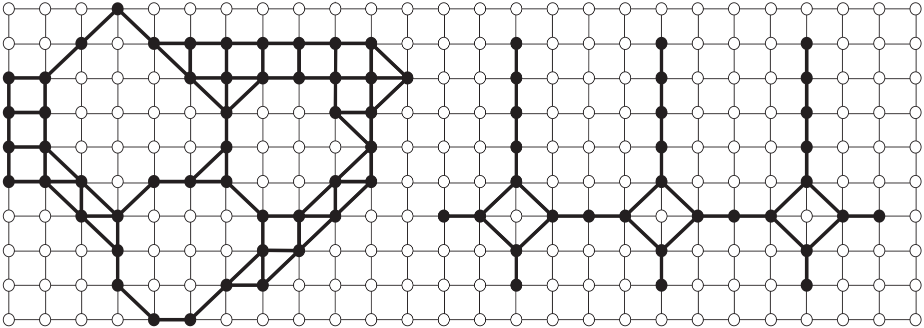
- $[S, \mathcal{G}]$ is called a digital topology in the grid cell model iff $S = \mathbb{C}_n^{(n)}$ $(n \geq 1)$ and $\mathcal{G}$ is a family of open sets that satisfy **T1** through **T3**, as well as the following:

  DT1: All connected sets are 0-connected
  DT2: All disconnected sets are $(n-1)$-disconnected
  DT3: The closure of any singleton (i.e., a 1-element set) is $(n-1)$-connected

- Up to homeomorphism, there is only one digital topology on $\mathbb{Z}^1$ (alternating topology)
- Up to homeomorphism, there are only two digital topologies on $\mathbb{Z}^2$ (grid point topology and grid cell topology)
- Up to homeomorphism, there are only five digital topologies on $\mathbb{Z}^3$
- Up to homeomorphism, there are only 24 digital topologies on $\mathbb{Z}^4$

Homeomorphy: Topological Equivalence of Binary Components

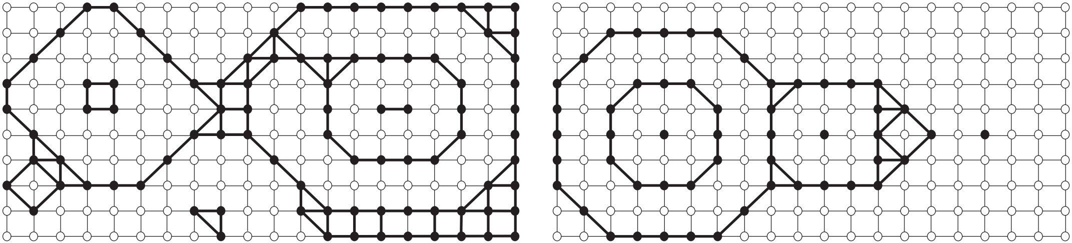
- Two components in an nD binary image are called topologically equivalent (homeomorphic) iff their geometric representations in the incidence grid are homeomorphic in $\mathbb{E}^n$



Two homeomorphic foreground components when considering $(8, 4)$-adjacency

Isotopy: Topological Equivalence of Binary Images

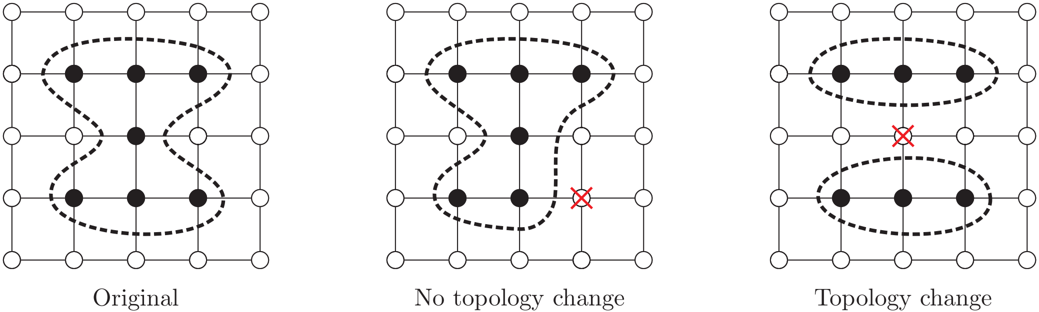
- Two subsets L and M of a topological space S are called isotopic iff there exists a homeomorphism Φ on S such that $\Phi(L) = M$
- Isotopy is a stronger concept than homeomorphy
- Two nD binary images are called topologically equivalent (isotopic) iff their geometric representations in the incidence grid are isotopic in $\mathbb{E}^n$
- Two nD binary images are isotopic iff their rooted adjacency trees are isomorphic



Are these two binary images isotopic when considering $(8, 4)$-adjacency?

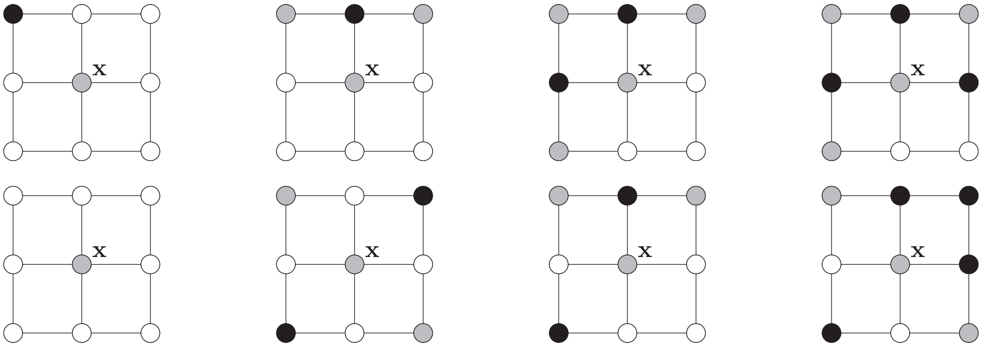
Simple Point Concept: Topology-Preserving Deformations

- Two isotopic binary images can be transformed one to another using topology-preserving deformations
- Topology-preserving deformations consist in switching the binary image values from foreground to background or from background to foreground at simple points



Simple Points in 2D Binary Images

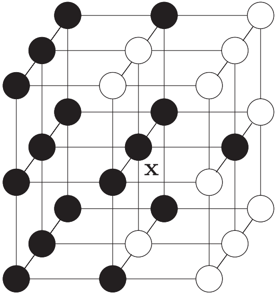
- Let *I* be a 2D binary image defined over a grid $\mathbb{G}$, and (α_1, α_2) be a pair of topologically compatible α-adjacencies
- A grid point $x \in \mathbb{G}$ is called (α_1, α_2)-simple iff it is α_1-adjacent to exactly one α_1-connected foreground component in $A_8(x)$ and α_2-adjacent to exactly one α_2-connected background component in $A_8(x)$
- Simple points can efficiently be detected using a precomputed LUT over all 256 possible configurations

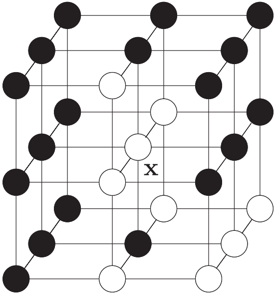


Top: $(8, 4)$-simple points; Bottom: Points that are not $(8, 4)$-simple; Gray indicates an arbitrary binary value

Simple Points in 3D Binary Images

- The extension of 2D simple point characterization is not straightforward because tunnels can be created or destroyed
- The detection of 3D simple points is carried out by calculating two topological numbers using BFS in 26-adjacency sets



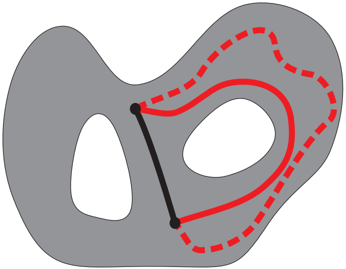


Left: Switching *x* from foreground to background creates a tunnel

Right: Switching *x* from background to foreground destroys a tunnel

Homotopy: Simply Connected Sets

- Let M be a subset of a topological space $[S, \mathcal{G}]$
- Homotopy provides a precise definition of the topological structure of M
- M is called path-connected iff, for any $p, q \in M$, there exists a parametrized path from p to q contained in M
- Two parametrized paths with the same fixed endpoints and contained in M are called homotopic iff one can continuously be transformed in M into another
- M is called simply connected iff it is path-connected and, for all $p, q \in M$, all parametrized paths from p to q contained in M are homotopic
- If M is simply connected, all loops contained in M are contractible in M into a single point



Take-Home Messages

- Homeomorphisms preserve topological invariants of topological spaces
- The frontier of a set is the border of its closure
- Isotopic binary images have isomorphic region adjacency trees
- Switching values of binary images at simple points does not change their topology
- Simply connected sets are path-connected, and all their loops are contractible in them into a single point