PA170 Digital Geometry

Lecture 07: Topological Characterization of Curves and Surfaces

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Recap: Common Subfields of Topology

Point Set Topology

- It considers local properties of spaces, and is closely related to analysis
- It generalizes the concept of continuity to define topological spaces, in which limits of sequences can be considered

Combinatorial Topology

- It is the oldest branch of topology, which dates back to Euler
- It considers the global properties of spaces, built up from a network of vertices, edges, and faces

Algebraic Topology

- It also considers the global properties of spaces, and uses algebraic objects such as groups or rings to answer topological questions
- It converts topological problems into algebraic ones that are hopefully easier to solve

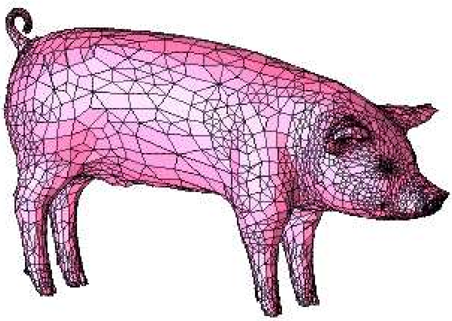
Differential Topology

- It considers spaces with some kind of smoothness associated to each point (e.g., a square and a circle are not differentiably equivalent to each other)
- It is useful for studying properties of vector fields, such as magnetic or electric fields

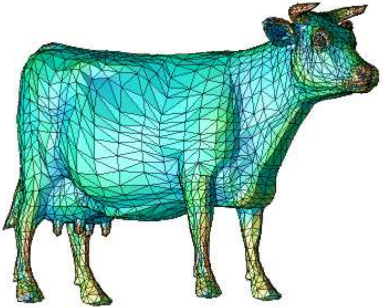
INTRODUCTION TO COMBINATORIAL TOPOLOGY

Motivation: Combinatorial Topology

- Combinatorial topology studies the topological properties of sets represented as complexes of small parts
- The topological properties are derived from these complexes



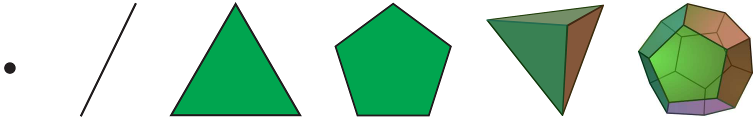
A Euclidean complex



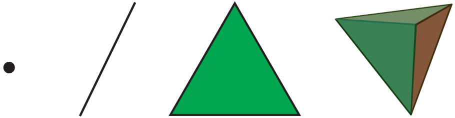
A simplicial complex

Common Types of Complexes

- Euclidean complexes consist of convex cells

Examples: 

- Simplicial complexes consist of simplices

All n-simplices ($0 \leq n \leq 3$): 

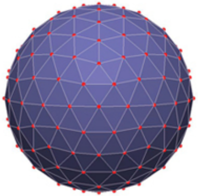
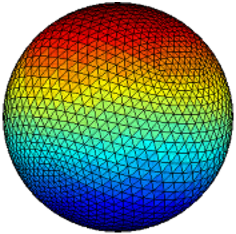
- Simplicial complexes are special cases of Euclidean complexes

Euclidean Complexes: Definition

- Let $M \subseteq \mathbb{E}^n$ be the union of a finite number of convex cells
- A Euclidean complex is a partition S of M into a nonempty finite set of convex cells with the following two properties:

  EC1: If p is a cell of S and q is a side of p, q is a cell of S
  EC2: The intersection of two cells of S is either empty or a side of both cells

- A finite Euclidean complex that contains only triangles, line segments, and points is called a triangulation

CURVES

Simple Closed Curves: Definitions

- A simple closed curve γ splits the plane into two open components. One component is bounded and the other component is unbounded, with γ being the frontier between these components

Parametric Definition

- γ is a set of points $\{(x, y) : \phi(t) = (x, y) \wedge a \leq t \leq b\}$ where $\phi : [a, b] \rightarrow \mathbb{R}^2$ is a continuous mapping the image of which is homeomorphic to a unit circle

Implicit Definition

- γ is a set of points $\{(x, y) : f(x, y) = 0\}$ satisfying an equation $f(x, y) = 0$

Topological Definition

- γ is a one-dimensional continuum (a nonempty, compact, and topologically connected subset of a topological space) in $\mathbb{E}^n$

Curves and Arcs: Terminology

- A simple curve is a curve in which every point p has branching index 2
- A simple arc is either a curve in which every point p has branching index 2 except for its two endpoints, which have branching index 1, or a simple curve with one of its points labeled as an endpoint

- A regular point of a curve has branching index 2 and is not an endpoint
- A branch point has branching index 3 or greater
- A singular point is either an endpoint or a branch point

- An elementary curve is the union of a finite number of simple arcs, each pair of which have at most a finite number of points in common
- Elementary curves can be approximated by polygonal chains

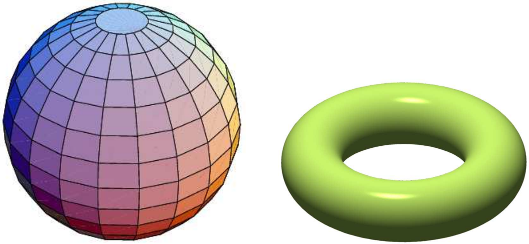
Topological Characterization of Elementary Curves

- An elementary curve γ can be partitioned into a one-dimensional geometric complex S that consists of α_1 simple arcs (1-cells), α_0 endpoints (0-cells), and β_0 components
- The Euler characteristic $\chi(S)$ of S is defined as $\chi(S) = \alpha_0 - \alpha_1$; $\chi(S)$ is preserved for any partition of γ
- The connectivity $\beta_1(S)$ of S is given as $\beta_1(S) = \beta_0 - \alpha_0 + \alpha_1$; $\beta_1(S)$ is equal to the number of atomic cycles of S
- Both the Euler characteristic and connectivity are topological invariants

SURFACES

Manifolds

- Let $[S, \mathcal{G}]$ be a topological space and $p \in S$
- Any subset of S that contains an open superset of p is called a topological neighborhood of p
- $[S, \mathcal{G}]$ is called an n-manifold if every $p \in S$ has a topological neighborhood in S, which is homeomorphic to an open n-sphere (near each point resembles $\mathbb{E}^n$)
- An n-manifold is called hole-free iff it is compact (closed and bounded)
- The surfaces of a ball and of a torus are examples of hole-free 2-manifolds



Simple Closed Surfaces: Definitions

- A simple closed surface σ splits $\mathbb{E}^3$ into two open components. One component is bounded and the other component is unbounded, with σ being the frontier between these components

Parametric Definition

- σ is a set of points $\sigma = \{(x, y, z) : \phi(s, t) = (x, y, z) \wedge a \leq s, t \leq b\}$ where $\phi : [a, b] \times [a, b] \rightarrow \mathbb{R}^3$ is a continuous mapping the image of which is homeomorphic to a unit sphere

Implicit Definition

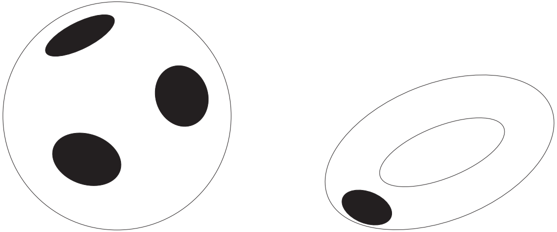
- σ is a set of points $\{(x, y, z) : f(x, y, z) = 0\}$ satisfying an equation $f(x, y, z) = 0$

Topological Definition

- σ is a hole-free surface (a hole-free 2-manifold)

Surfaces with Frontiers: Definition

- A surface S is called a surface with frontiers iff S is homeomorphic to a polyhedral surface and can be partitioned into two nonempty subsets S° and ϑS such that every $p \in S^\circ$ has a topological neighborhood in S, which is homeomorphic to an open disk, and every $p \in \vartheta S$ has a topological neighborhood in S, which is homeomorphic to the union of the interior of a triangle and one of its sides (without endpoints) where p is mapped onto that side
- The points of S° are called interior points of S, and the points of ϑS are called frontier points of S
- The number of frontiers is a topological invariant



A surface with three frontiers　　A surface with one frontier

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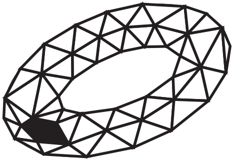
A surface with three frontiers



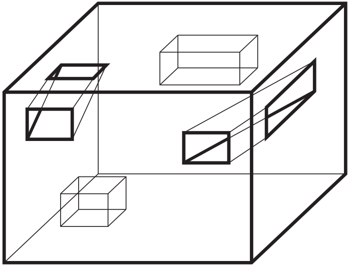
A surface with one frontier

Topological Characterization of 2D Euclidean Complexes

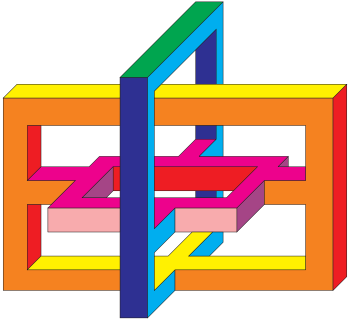
- Let S be a 2D Euclidean complex that consists of α_i i-cells ($0 \le i \le 2$)
- The Euler characteristic of S is defined as $\chi(S) = \alpha_0 - \alpha_1 + \alpha_2$
- If S is a simple polyhedron, $\chi(S) = 2$ (Descartes & Euler)
- In 1812, Lhuilier incorrectly derived the following formula:

$$\alpha_0 - \alpha_1 + \alpha_2 = 2(c - t + 1) + p$$

where c is the number of cavities, t is the number of tunnels, and p is the number of polygons ("tunnel exits") on the polyhedron faces



Tunnels and cavities



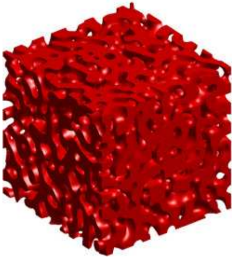
What are tunnels?

Betti Numbers

- Betti numbers β_i ($0 \leq i \leq n$) are topological invariants, which extend the polyhedral formula to n-dimensional spaces (the Poincaré formula):

$$\chi(\cdot) = \sum_{i=0}^{n} (-1)^i \cdot \beta_i$$

- Informally, β_0 is the number of connected components, β_1 is the number of tunnels, and $\beta_0 + \beta_2$ is the number of closed surfaces (so that there are β_2 cavities)
- Formally, β_i is the rank of the i-th homology group of the particular topological space, and can be algorithmically calculated

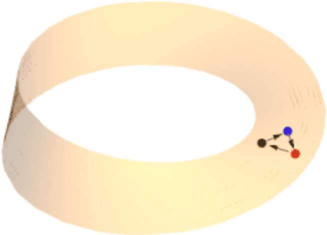


$\beta_0 = 1$, $\beta_1 = 1059$, $\beta_2 = 0$

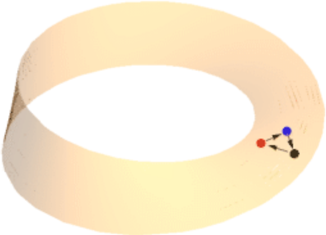
The Orientability of Surfaces

- An oriented triangle is a triangle with a direction on its frontier (e.g., clockwise or counterclockwise), which is called the orientation of that triangle
- Two triangles are called coherently oriented if they induce opposite orientations on their common side
- A triangulation of a surface is called orientable iff it is possible to orient all of the triangles in such a way that every pair of triangles with a common side is coherently oriented; otherwise it is called nonorientable
- All triangulations of the same surface are either orientable or nonorientable
- A surface is called orientable iff it has an orientable triangulation
- The orientability of a surface is a topological invariant

A Möbius strip



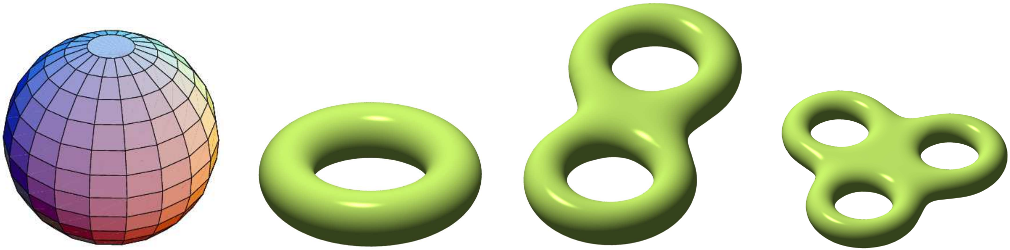
A starting configuration



The configuration after one loop

The Genus of Orientable Surfaces

- Let S be an orientable surface. The genus $g(S)$ of S is the number of handles of S
- It can be shown that $\chi(S) = 2 - 2g(S)$
- The genus is a topological invariant



Genus 0 Genus 1 Genus 2 Genus 3

Source: MathWorld, Wikipedia

Take-Home Messages

- A simple closed curve is defined as a one-dimensional continuum
- The topology of elementary curves can be characterized by the Euler characteristic and connectivity
- A simple closed surface is defined as a hole-free 2-manifold
- The topology of surfaces is uniquely determined by the number of frontiers, orientability, and Euler characteristic
- Betti numbers can correctly describe tunnels and cavities in surfaces
- A Möbius strip is a popular example of a nonorientable surface