PA170 Digital Geometry

Lecture 08: Digital Straightness and Convexity

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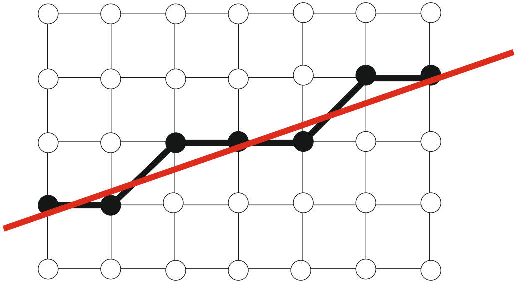
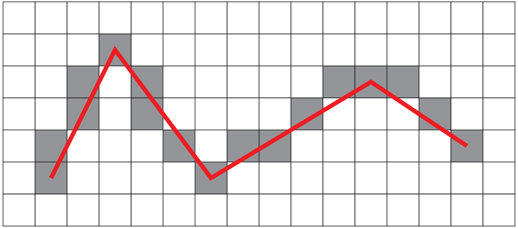
Autumn 2023

Motivation: Straightness in 2D Digital Spaces

- A digital arc is considered straight if it corresponds to the digitization of a straight line segment
- The recognition of digital straight line segments and decomposition of digital arcs into such segments play crucial roles in many applications such as the estimation of curve length or the simplification of shapes



A straight line and its digitization



Four straight line segments and their digitization

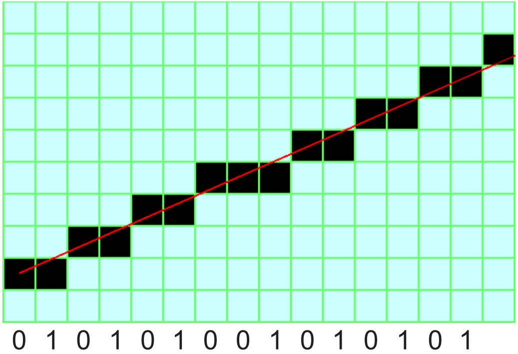
DIGITAL RAYS

Digital Rays as Binary Words

- Let $\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : 0 \leq x < \infty, 0 \leq \alpha \leq 1\}$ be a real ray with the slope α and intercept β. Its grid-intersection digitization $R(\gamma_{\alpha,\beta})$ is called a digital ray, and it corresponds to a sequence of grid points $(n, I_n) \in \mathbb{Z}^2$:

$$R(\gamma_{\alpha,\beta}) = I_{\alpha,\beta} = \{(n, I_n) : n \geq 0 \wedge I_n = \lfloor \alpha n + \beta + 0.5 \rfloor\}$$

- Digital rays can be represented as binary words $i_{\alpha,\beta}$ over the $\{0, 1\}$ alphabet where $i_{\alpha,\beta}(n) = I_{n+1} - I_n, n \geq 0$, zeros correspond to horizontal steps, and ones correspond to diagonal steps



Word Theory Terminology and Notation

- A word defined over an alphabet A is a sequence of elements of A
- The set of all words defined over A is denoted as $A^\star$
- The set of all infinite words over A is denoted as A^∞

- A word $v \in A^\star$ is called a factor of a word $u \in A^\star$ iff there exist words $v_1 \in A^\star$ and $v_2 \in A^\star$ such that $u = v_1 v v_2$

- An integer $k \geq 1$ is called a period of a word $u = a_0 a_1 \ldots a_{n-1}$ if $a_i = a_{i+k}$ $(i = 0, \ldots, n-1-k)$
- An infinite word $w \in A^\infty$ is called periodic if it is of the form $w = v^\infty$ for some nonempty finite word $v \in A^\star$
- An infinite word $w \in A^\infty$ is called eventually periodic if it is of the form $w = uv^\infty$ for some nonempty finite words $u, v \in A^\star$
- An infinite word $w \in A^\star$ is called aperiodic if it is not eventually periodic

Digital Rays: Terminology and Theory

- Digital rays are right infinite binary words
- Digital straight lines (DSLs) are infinite binary words in both directions
- Digital straight line segments (DSSs) are finite binary words
- DSSs are nonempty finite factors of digital rays

- A finite or infinite 8-arc is called irreducible iff its set of grid points does not remain 8-connected after removing a nonendpoint from it
- A digital ray is called rational if α is rational and irrational if α is irrational

- **Connectivity**: A digital ray is an irreducible 8-arc
- **Self-similarity**: For irrational α, $I_{\alpha,\beta}$ uniquely determines both α and β. For rational α, $I_{\alpha,\beta}$ uniquely determines α, and β is determined up to an interval
- **Periodicity**: Rational digital rays are periodic and irrational digital rays are aperiodic

CHARACTERIZATION OF DIGITAL STRAIGHT LINE SEGMENTS

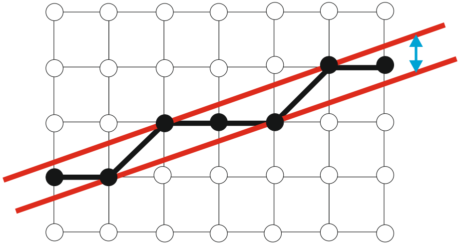
Offline and Online DSS Recognition Algorithms

- Many linear DSS recognition algorithms have been introduced

- Offline algorithms decide whether a finite binary word $u \in \{0, 1\}^\star$ is a DSS
- Online algorithms sequentially read a finite binary word $u \in \{0, 1\}^\star$ and determine the highest $k \geq 0$ such that $u(0), \ldots, u(k)$ is a DSS, but $u(0), \ldots, u(k+1)$ is not

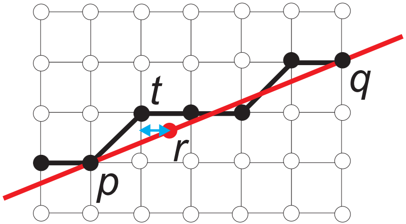
- Remark: Linear online algorithms produce digital arc decomposition in linear time, whereas linear offline algorithms carry out this decomposition in quadratic time.

- Let u be an 8-arc of length n and $G(u) = \{p_0, p_1, \ldots, p_{n-1}\}$ be the assigned set of grid points such that $p_0 = (0, 0)$ and u connects p_0 with p_{n-1} via a sequence of horizontal and diagonal steps through $p_1, \ldots, p_{n-2}$
- u is a DSS iff $G(u)$ lies on or between two parallel lines with a distance apart that is less 1 (measured in the vertical direction)



Chord Property

- Let u be an irreducible 8-arc of length n and $G(u) = \{p_0, p_1, \ldots, p_{n-1}\}$ be the assigned set of grid points such that $p_0 = (0, 0)$ and u connects p_0 with p_{n-1} via a sequence of horizontal and diagonal steps through $p_1, \ldots, p_{n-2}$
- $G(u)$ satisfies the chord property iff for any two different points $p, q \in G(u)$ and any point r on the real line segment between p and q, there exists a grid point $t \in G(u)$ such that $d_8(r, t) < 1$
- u is a DSS iff $G(u)$ satisfies the chord property



Syntactic Characterization: Preliminaries

- Let $u = (u(i))_{i \in I}$ ($I \subseteq \mathbb{Z}$) be a word over $\mathbb{N}$
- A digit $k \in \mathbb{N}$ is called singular in u iff k appears in u and, for all $i \in I$ such that $i - 1$ and $i + 1$ are in I, if $u(i) = k$, $u(i-1) \neq k$ and $u(i+1) \neq k$
- A digit $k \in \mathbb{N}$ is called nonsingular in u iff k appears in u and is not singular in u
- u is reducible iff u contains no singular digit or any factor of u that contains only nonsingular digits is finite
- Let u be reducible, and let $R(u)$ be one the following:
  - The length of u if u is finite and contains no singular digit; or
  - The word that results from u by replacing all of its factors of nonsingular digits in u that are between two singular digits in u by their run lengths, and by deleting all other digits from u; or
  - The digit d if $u = d^\infty$
- Recursive application of this reduction operation R produces a sequence of reducible words $u_0, u_1, \ldots$ where $u_0 = u$ and $u_{i+1} = R(u_i)$, $i \geq 0$

What is the sequence of reducible words after recursive application of R on $u = 1101110111011101111011101110111101110111101110111011110111011101110111101110111011101110111101110111011110111$?

Syntactic characterization of DSLs and DSSs

- A two-sided infinite 8-arc u is a DSL iff $u_0 = u, u_1, \ldots$ (where $u_{i+1} = R(u_i)$, $i \geq 0$) are reducible words and satisfy the following two conditions:
  - **L1** There are at most two different digits d and e in u_i, and, if there are two, $|d - e| = 1$
  - **L2** If there are two different digits in u_i, at least one of them is singular.

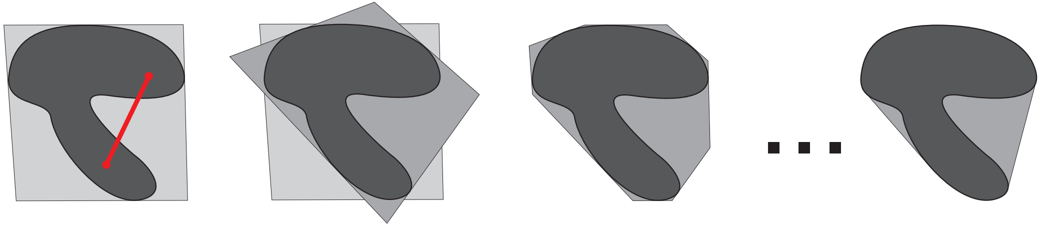
- Let u be a finite 8-arc, and $l(u)$ and $r(u)$ be the run lengths of nonsingular digits to the left of the first singular digit and to the right of the last singular digit in u. u is a DSS iff $u = u_0$ satisfies **L1** and **L2**, and any sequence $u_i = R(u_{i-1})$ satisfies **L1** and **L2** as well as the following:
  - **S1** If u_i contains only one digit d or two different digits d and $d+1$, $l(u_{i-1}) \leq d+1$ and $r(u_{i-1}) \leq d+1$
  - **S2** If u_i contains two different digits d and $d+1$ and d is nonsingular in u_i, u_i starts with d if $l(u_{i-1}) = d+1$ and ends with d if $r(u_{i-1}) = d+1$

Is $u =$ 1101110111011101111011101110111101110111011110111011101111011101110111011110111011101111 0111 a DSS?

DIGITAL CONVEXITY

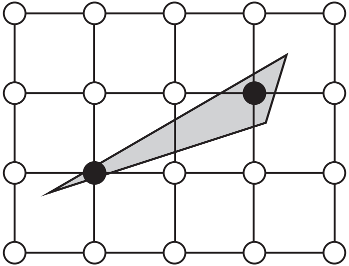
Convex Sets and Convex Hulls

- $M \subseteq S$ is called convex if, for any distinct $p, q \in M$, the straight line segment between p and q is contained in M
- Any intersection of convex subsets of S is a convex set
- The intersection $C(M)$ of all of the convex subsets of S that contain M is called the convex hull of M

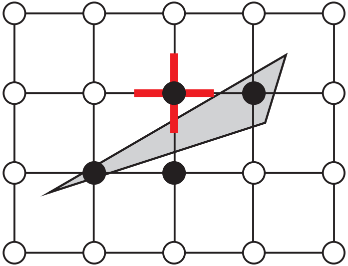


Digitally Convex Sets

- A digital set is convex if it is obtained as the digitization of a convex set
- The Gauss digitization of a convex set $M \subseteq \mathbb{R}^2$ may not be 8-connected
- The cross digitization of a convex set $M \subseteq \mathbb{R}^2$ is always simply 8-connected
- A set $\emptyset \neq S \subseteq \mathbb{Z}^2$ is called digitally convex iff there exists a convex set $M \subseteq \mathbb{R}^2$, the cross digitization of which is S



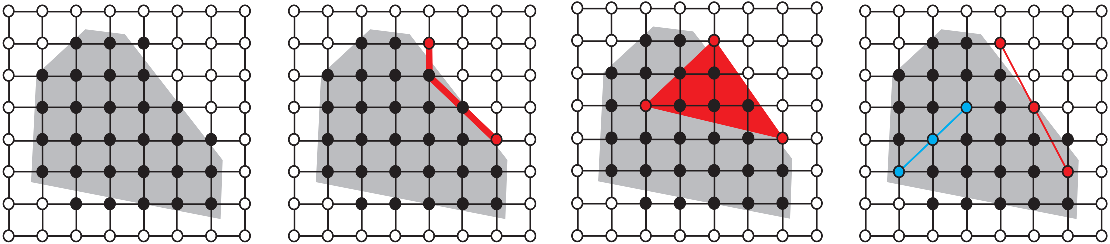
Gauss digitization of a convex set



Cross digitization of a convex set

Digitally Convex Sets: Properties

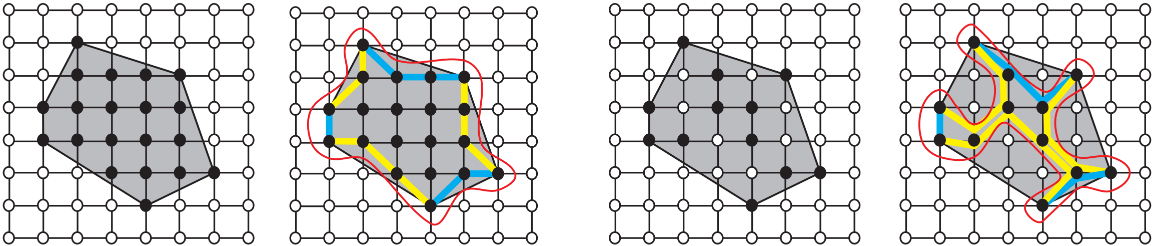
- A finite set $M \subseteq \mathbb{Z}^2$ is digitally convex iff any of the following is true:
  - M satisfies the chord property
  - For all $p, q \in M$, at least one DSS with p and q as endpoints is contained in M
  - For $p, q, r \in M$, all of the grid points in the triangle pqr are in M
  - Any grid point on the real line segment between two grid points of M is also in M



A digitally convex set — The 2nd property — The 3rd property — The 4th property

Border Segments: Digital Convexity Test

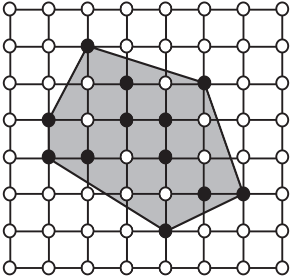
- The Euclidean convex hull $C(M)$ of any $M \subseteq \mathbb{Z}^2$ is a convex polygon with vertices that are all 4-border pixels of M
- If M is simply 8-connected, 4-border tracing visits all of the 4-border pixels of M sequentially
- The vertices of $C(M)$ partition the sequence of border pixels into subsequences that start and end at successive vertices of $C(M)$
- These subsequences are called border segments of M



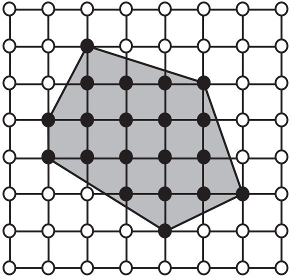
A digitally convex set — Its border segments — A digitally nonconvex set — Its border segments

Digital convexity test: A finite proper simply 8-connected subset M of $\mathbb{Z}^2$ is digitally convex iff every border segment of M is a DSS

- S is called a digitally convex completion of $M \subseteq \mathbb{Z}^2$ iff S contains M and is digitally convex, and $S \setminus U$ is not digitally convex for any $U \subseteq S \setminus M$



A digitally nonconvex set



Its convex completion

Take-Home Messages

- Digital rays can be represented as right infinite binary words
- Digital straight lines can be represented as two-sided infinite binary words
- Digital straight line segments are nonempty finite factors of digital rays
- Digital straight line segments can be characterized using the tangential lines, chord property, or syntactic analysis
- Digitally convex sets are the cross digitization counterparts of real convex sets
- Border segments of simply 8-connected digital sets can efficiently be exploited to check whether these sets are digitally convex