Rotations and quaternions

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PA199

Outline

- Rotation matrix

- Euler angles
- Tait-Bryan angles

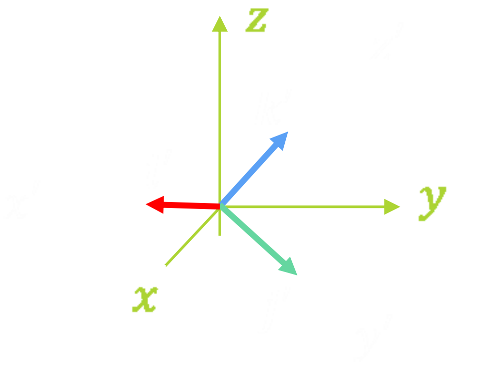
- Axis-angle representation

- Quaternions
  - Rotations via quaternions
  - Quaternion derivative

Rotation matrix

- Well know topic from computer graphics courses
  → We only discuss relation between basis vectors and rotation matrix.
- Let i', j', k' be orthonormal basis vectors of a coordinate system inside the world coordinate system.
- Then, the orientation of the coordinate system is represented by the rotation matrix:

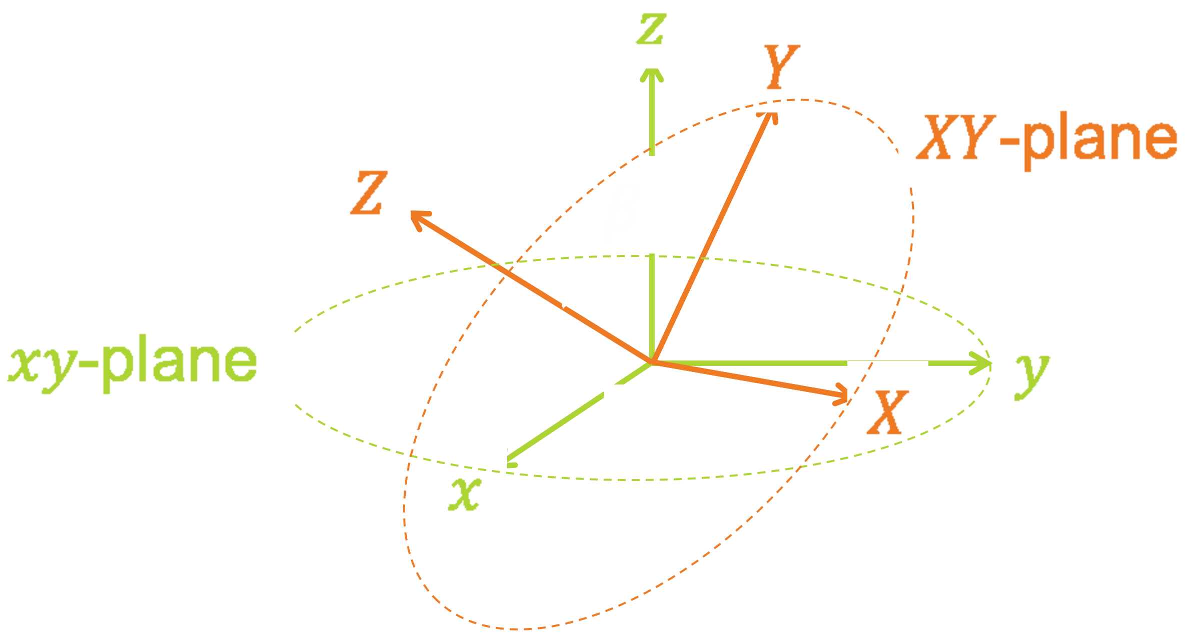
$$R = \begin{pmatrix} i'_x & j'_x & k'_x \\ i'_y & j'_y & k'_y \\ i'_z & j'_z & k'_z \end{pmatrix}$$



- R transforms vectors "to world space".
- $R^{-1} = R^T$ transforms vectors "from world space".

Euler angles

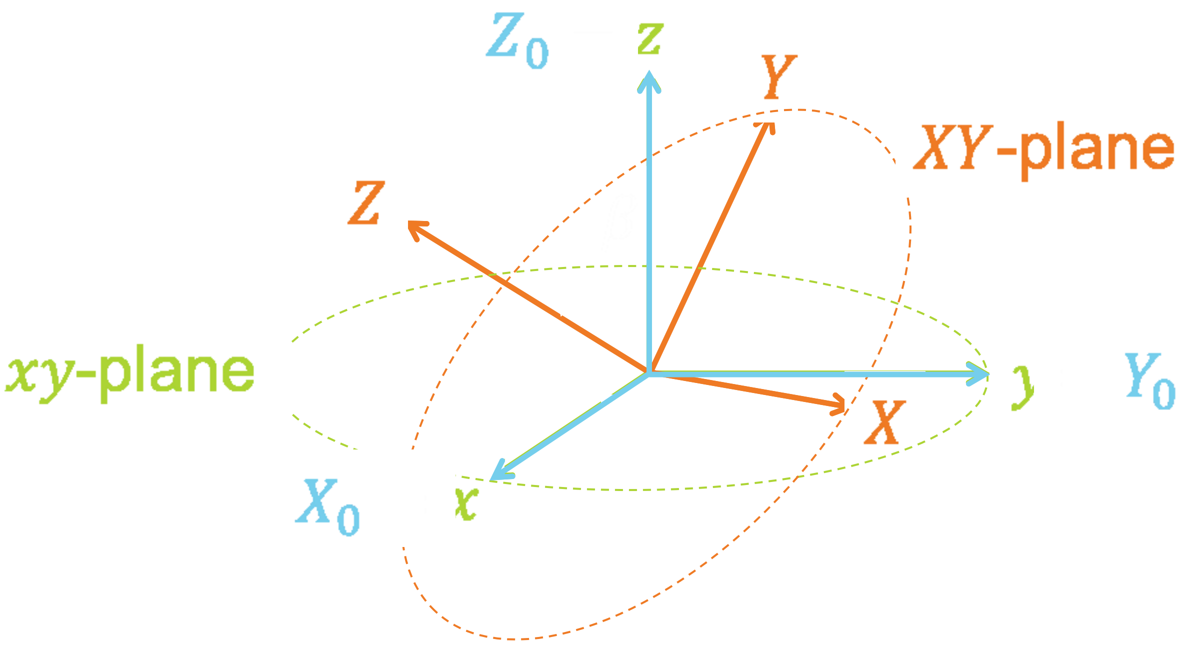
- Three rotations are always sufficient to transform a source frame xyz to a target one XYZ:
  - $\alpha \in (0, 2\pi)$
  - $\beta \in (0, \pi)$
  - $\gamma \in (0, 2\pi)$
- Actual rotations:



When planes xy and XY are equal, then the statement is clearly true

Euler angles

- Three rotations are always sufficient to transform a space from xyz to a target one XYZ:
  - $\alpha \in (0, 2\pi)$
  - $\beta \in (0, \pi)$
  - $\gamma \in (0, 2\pi)$
- Actual rotations:
  - Start with frame $X_0Y_0Z_0 = xyz$.
  - Rotate $X_0Y_0Z_0$ about Z_0 by α.



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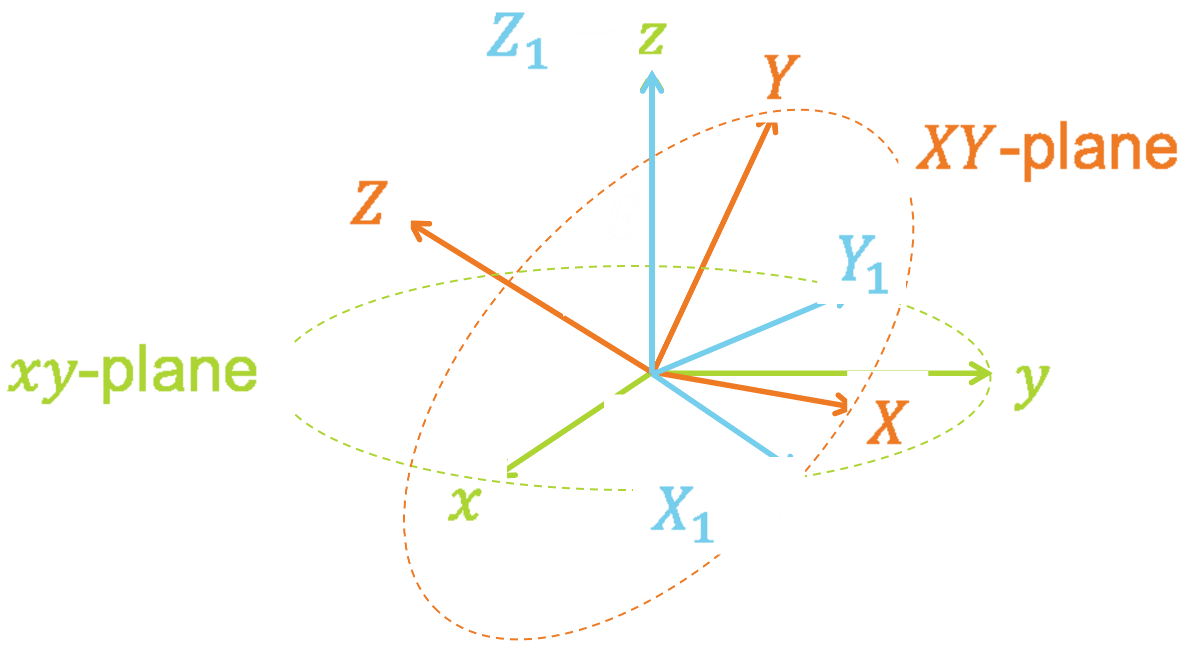
  - Rotate $X_0Y_0Z_0$ about Z_0 by α.
  - Rotate $X_1Y_1Z_1$ about X_1 by β.



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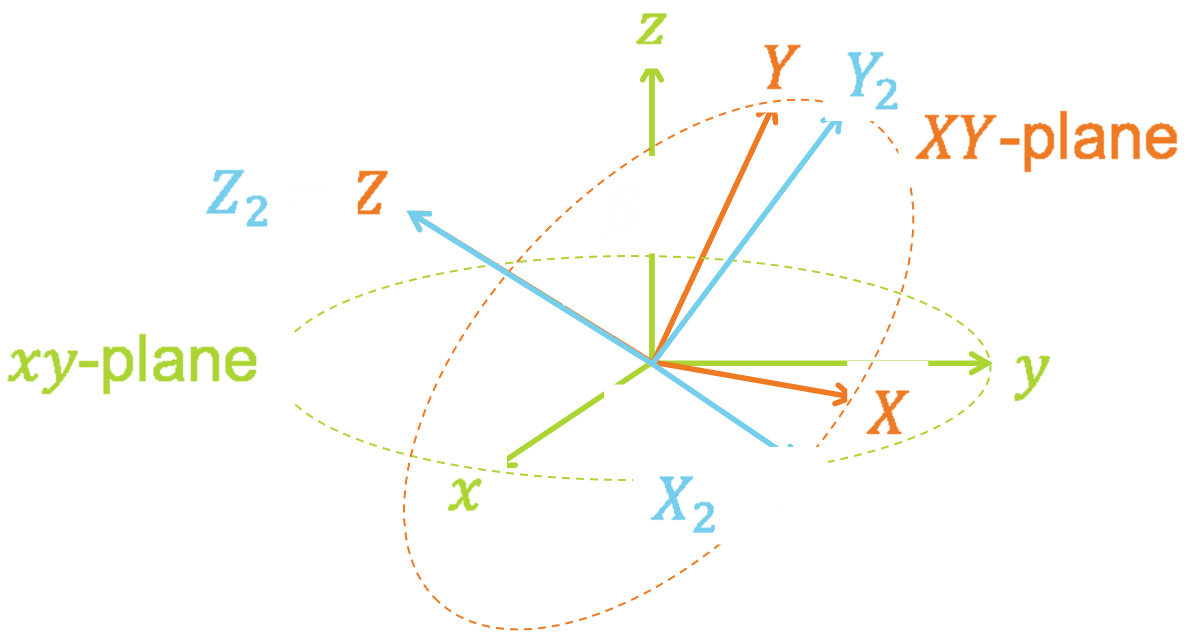
  - Rotate $X_1Y_1Z_1$ about X_1 by β.
  - Rotate $X_2Y_2Z_2$ about Z_2 by γ.



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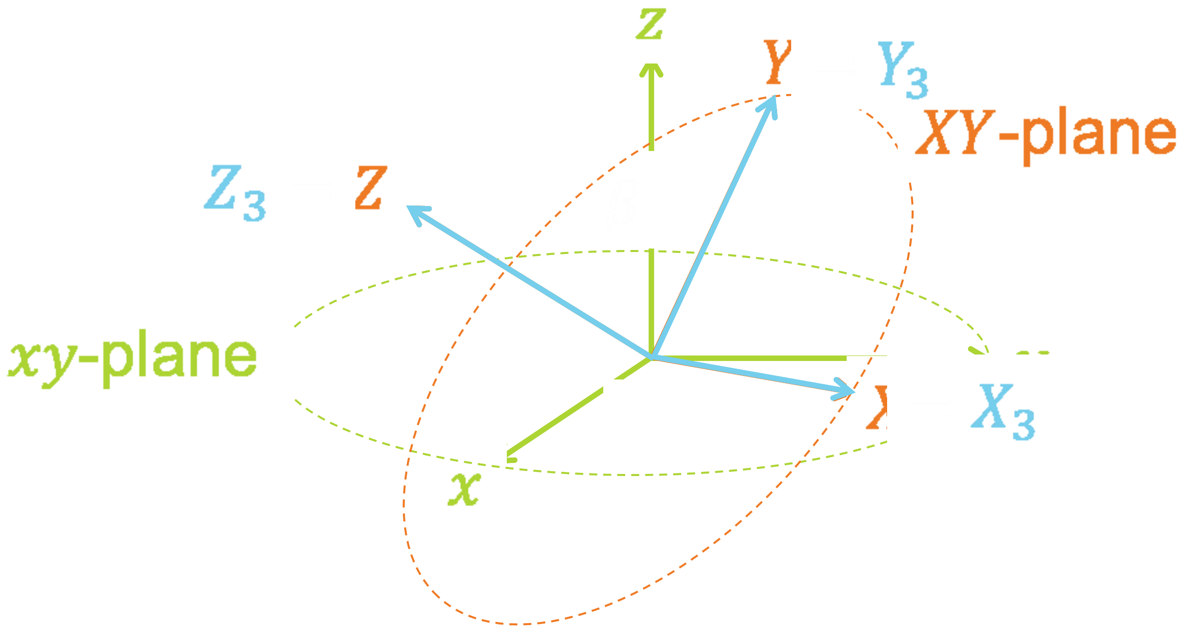


When planes xy and XY are equal, then the statement is clearly true

Euler angles

- Let $R(\varphi, a)$ denotes a rotation matrix about an axis a by an angle φ.
- So, our rotations can be expressed by matrices:
  - $R(\alpha, Z_0)$
  - $R(\beta, X_1)$
  - $R(\gamma, Z_2)$
- We compose them by the matrix multiplication:

$$R(\gamma, Z_2)R(\beta, X_1)R(\alpha, Z_0)$$

- Here we work with Z-X-Z convention. But there are 5 more:
  - X-Y-X, X-Z-X, Y-X-Y, Y-Z-Y, and Z-Y-Z.
  - We can choose any of the conventions we want.
  - Observation: 1st and 3rd rotation axes are the same.

Euler angles

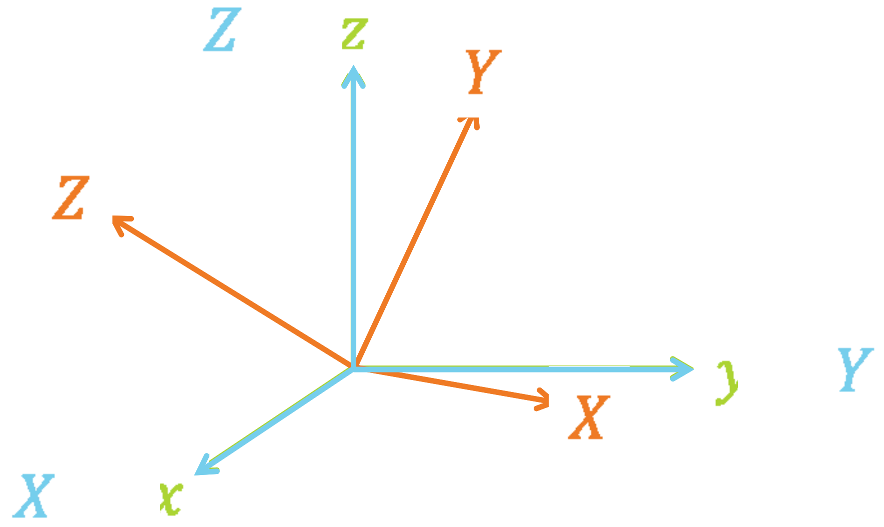
- The rotations $R(\alpha, Z_0)$, $R(\beta, X_1)$, $R(\gamma, Z_2)$ about the axes of the rotated (target) frame XYZ are called intrinsic.

- A practical disadvantage of intrinsic rotations is that some of rotations are about arbitrary oriented axis.
  - In CG courses we only learned how to build rotation matrices for fixed axes x, y, z.
  - But axes X_1, Z_2 may be arbitrary (the axis Z_0 is OK since $Z_0 = z$).

- Fortunately, we can also transform a source frame xyz to a target one XYZ using extrinsic rotations $R(\alpha, z)$, $R(\beta, x)$, $R(\gamma, z)$.
  - Let us figure out how to do that.

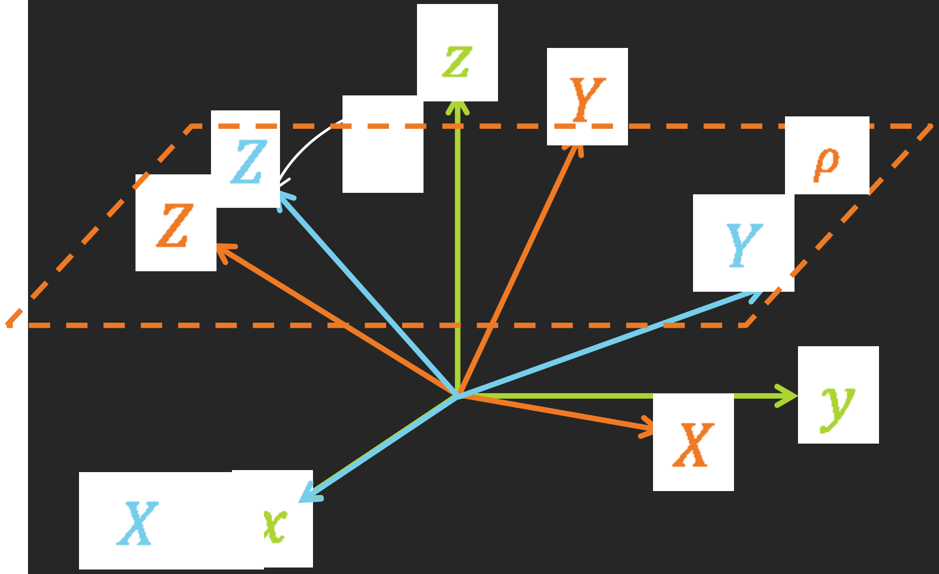
Euler angles

- Start with the XYZ aligned with xyz.



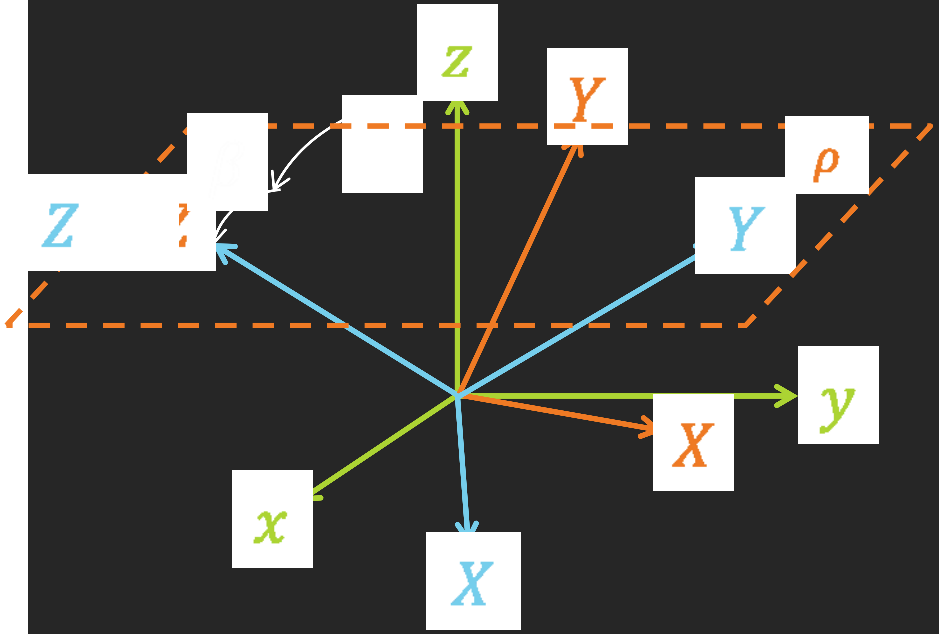
Euler angles

- Start with the XYZ aligned with xyz.
- Apply $R(\alpha, x)$ to rotate the tip of Z to the plane ρ.
  - ρ is parallel with xy plane and contains the tip of Z.



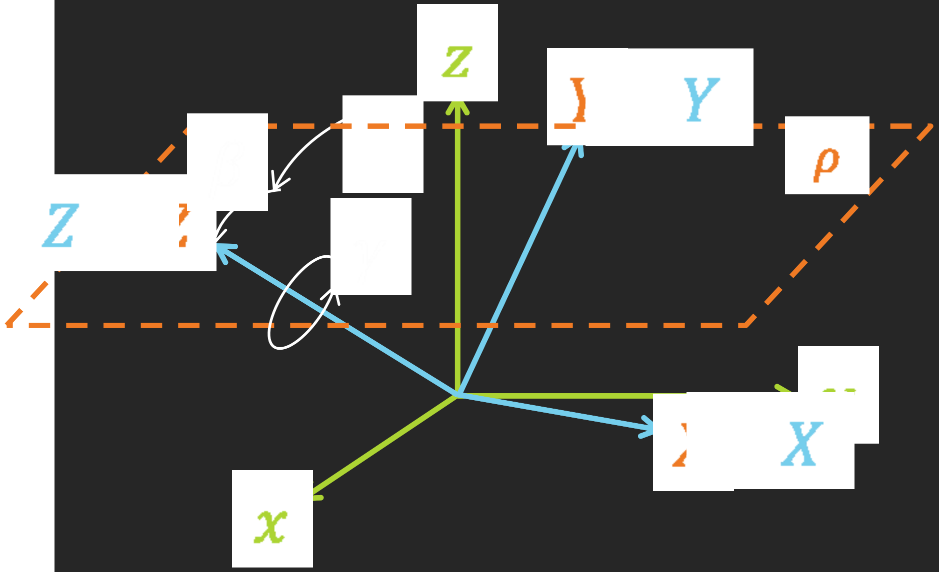
Euler angles

- Start with the XYZ aligned with xyz
- Apply $R(\alpha, x)$ to rotate the tip of Z to the plane ρ.
  - ρ is parallel with xy plane and contains the tip of Z.
- Apply $R(\beta, z)$ to rotate the tip of Z to the tip of Z.



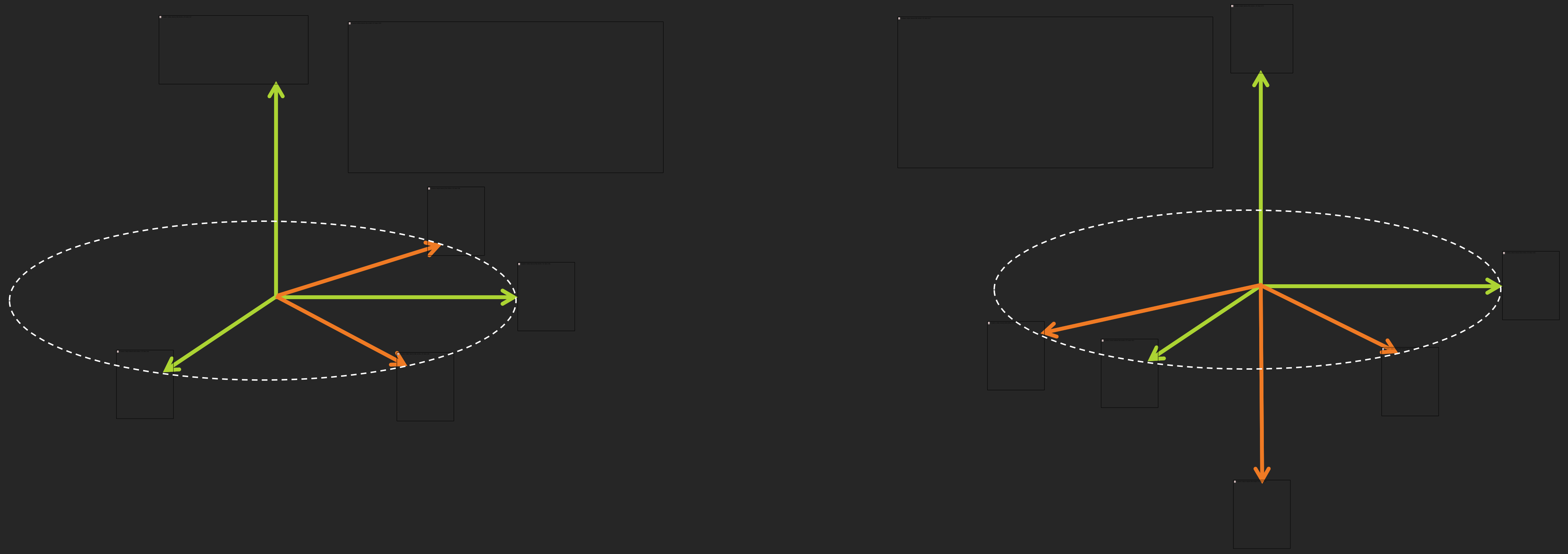
Euler angles

- Start with the XYZ aligned with xyz
- Apply $R(\alpha, x)$ to rotate the tip of Z to the plane ρ.
  - ρ is parallel with xy plane and contains the tip of Z.
- Apply $R(\beta, z)$ to rotate the tip of Z to the tip of Z.
- Apply the "twist" rotation $R(\gamma, Z)$ to align X with X and Y with Y.
  - But, this is not extrinsic rotation!
  - We can fix it by applying the twist $R(\gamma, z)$ as the first rotation.
  - The value of γ will be different since the other two rotations affect the twist too.



More intuition is in video: [5]

Euler angles: gimbal lock



Euler angles representation

- We can use 3 angles to express any orientation of an object in 3D space:

```
public class Orientation {
    float alpha;
    float beta;
    float gamma;
};
```

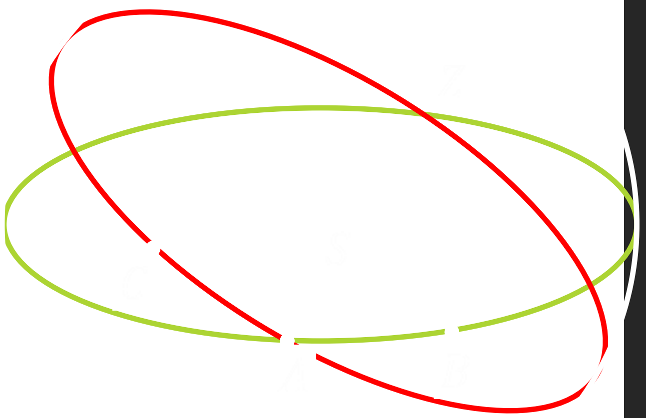
- Pros:
  - Low memory footprint.
  - Easy to understand.
- Cons:
  - Suffers from the gimbal lock.
  - Slow conversion to matrix representation (sin and cos for each angle).

Tait-Bryan angles

- Same as Euler angles, except that all three axes are different
- There are 6 possible conventions:
  - XYZ, XZY, YXZ, YZX, ZXY, and ZYX
- The line of nodes is different: It is an intersection of the xy plane and the plane orthogonal to the 3rd rotation axis of the convention.
- The angles α, β, γ are often called yaw, pitch, roll, respectively.

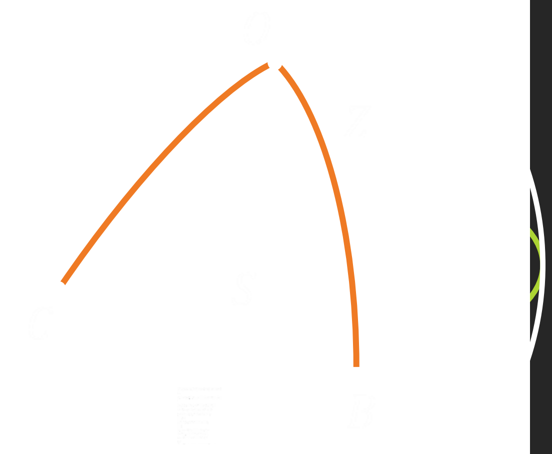
Axis-angle rotation

- Euler's rotation theorem (one of the versions): Any reconfiguration of an object in 3D space with one of its points fixed is equivalent to a single rotation about an axis passing through the fixed point.
- Proof:
  - We look for a rotation axis passing though S.
  - We "paint" a great circle (green) on the sphere in the initial position.
  - We rotate the sphere => We get the rotated green circle, which is depicted as red circle.
  - If the circles coincide, then the axis clearly exists. Otherwise, the circles intersect - two points A, Z.
  - A is on red circle => its pre-image B is on green one. A is on green circle => its post-image C is on red one.



Axis-angle rotation

- Construct a great circle (blue) passing through A, Z and bisecting the angle BAC.
- Find a point 0 on the blue circle s.t. the length of arcs $A0$ and $B0$ is the same.
- The length of the arc $A0$ must be equal to the length of the arc $C0$, because lengths of arcs AB and AC are the same and the blue circle is the bisector of the angle CAB.
  - ⇒ Triangles $CA0$ and $AB0$ on the sphere must be the same. Actually, $AB0$ becomes $CA0$ after the rotation.
  - ⇒ The point 0 lies on the reached rotation axis, because it does not move when rotating the triangles.
  - ⇒ $S0$ is the rotation axis and the arc length AB is the angle.



Axis-angle rotation

- Rodrigues' rotation formula: A vector $v \in R^3$ rotated about a unit axis $a \in R^3$ by an angle $\theta \in (0, 2\pi)$ is the vector:

$$\tilde{v} = \cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v.$$

  - Proof:

$v_a = (a \cdot v)a,$

$v_\perp = v - v_a = v - (a \cdot v)a,$

$v_\times = a \times v_\perp = a \times (v - v_a) = a \times v.$

Note: $|v_\times| = |v_\perp|$.

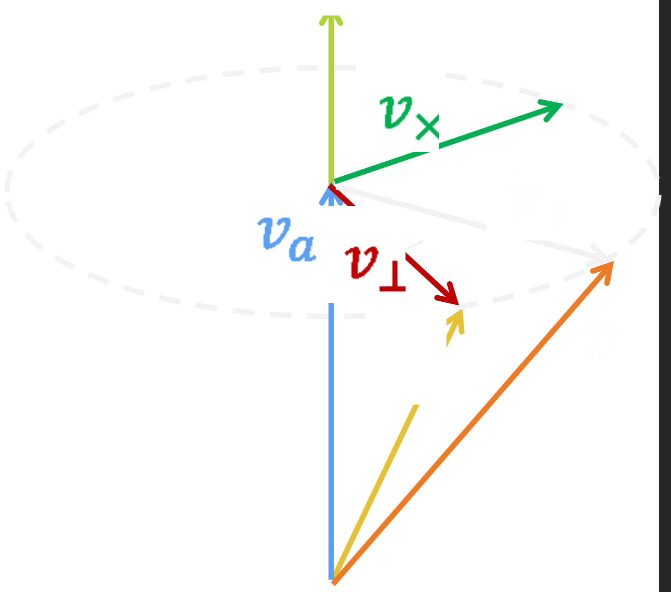
$\tilde{v}_\perp = \cos\theta\, v_\perp + \sin\theta\, v_\times.$

$\tilde{v} = v_a + \tilde{v}_\perp$

$= v_a + \cos\theta\, v_\perp + \sin\theta\, v_\times$

$= (a \cdot v)a + \cos\theta\,(v - (a \cdot v)a) + \sin\theta\, a \times v$

$= \cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v.$



Axis-Angle to rotation Matrix

- Vector triple product: $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$
  - If $|a| = 1$, then $a \times (a \times v) = (a \cdot v)a - (a \cdot a)v = (a \cdot v)a - v$
- Matrix representation of the cross product:

$$a \times v = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} v = [\![a]\!]v$$

- Matrix representation of the axis angle:

$$\begin{aligned} v' &= \cos\theta\, v + (1-\cos\theta)(a \cdot v)a + \sin\theta\, a \times v \\ &= \cos\theta\, v + (1-\cos\theta)(a \times (a \times v) + v) + \sin\theta\, a \times v \\ &= v + (1-\cos\theta)\, a \times (a \times v) + \sin\theta\, a \times v \\ &= v + (1-\cos\theta)[\![a]\!]^2 v + \sin\theta\, [\![a]\!]v \\ &= (I + (1-\cos\theta)[\![a]\!]^2 + \sin\theta\, [\![a]\!])v \\ &= R(\theta, a)v \end{aligned}$$

Linear interpolation (lerp)

- Given two axis-angle rotations $R(\varphi, \alpha)$ and $R(\psi, \beta)$, the linearly interpolated rotation is then $R\left((1-t)\varphi + t\psi, \frac{(1-t)\alpha + t\beta}{|(1-t)\alpha + t\beta|}\right), t \in (0,1)$.
- Technical issues related to $|(1-t)\alpha + t\beta|$:
  - Slow – we must compute the square root.
  - The result is not defined when =0.
- Problem: The velocity is not constant (increases and decreases).
  $\Rightarrow$ Visible visual artefact – we prefer uniform blending between rotations.
- Can we do better?
  - Yes, use spherical linear interpolation.

Spherical linear interpolation (slerp)

- Given two linearly independent unit vectors u, v and a parameter $t \in (0,1)$, find a unit vector $w = \alpha u + \beta v$ s.t. $\alpha, \beta > 0$ and angle between u, w is $t\theta$, where θ is the angle between u, v.
- $v_\perp = \frac{v - \cos\theta u}{\sqrt{(v - \cos\theta u)(v - \cos\theta u)}} = \frac{v - \cos\theta u}{\sqrt{1 - \cos^2\theta}} = \frac{v - \cos\theta u}{\sin\theta}$
- $w = \cos t\theta\, u + \sin t\theta \frac{v - \cos\theta u}{\sin\theta}$

$$= \left(\cos t\theta - \frac{\sin t\theta \cos\theta}{\sin\theta}\right) u + \frac{\sin t\theta}{\sin\theta} v$$

$$= \frac{\cos t\theta \sin\theta - \sin t\theta \cos\theta}{\sin\theta} u + \frac{\sin t\theta}{\sin\theta} v$$

$$= \frac{\sin(1-t)\theta}{\sin\theta} u + \frac{\sin t\theta}{\sin\theta} v.$$

- Given two axis-angle rotations $R(\varphi, a)$ and $R(\psi, b)$, the interpolated rotation is then $R\left((1-t)\varphi + t\psi, \frac{\sin(1-t)\theta}{\sin\theta} a + \frac{\sin t\theta}{\sin\theta} b\right)$

Axis-angle representation

- We can use axis-angle to express any orientation of object in 3D space.

```
public class Orientation {
    float angle;
    Vector3 unitAxis;
};
```

- Pros:
  - Fast conversion to matrix representation (sine and cosine for one angle).
  - We can use lerp and slerp.
  - Easy to understand.
  - Low memory footprint.
- Cons:
  - Complicated composition of rotations (often solved via other rep.).

Quaternions

- Let a, b are real numbers and $i = \sqrt{-1}$ be an imaginary unit. Then

$$a + bi$$

is a complex number (constructed by the pairing process).

- Let $a + bi, c + di$ are complex numbers and $j = \sqrt{-1}$ be an imaginary unit, $i \neq j$. Then

$$(a + bi) + (c + di)j =$$
$$a + bi + cj + dij =$$
$$a + bi + cj + dk$$

where $k = ij$, is a quaternion.

- $k = \sqrt{-1}$ is another unique imaginary unit, i.e., $k \neq i, k \neq j$.
- Relations between imaginary units:

$$ij = k, \quad jk = i, \quad ki = j,$$
$$ji = -k, \quad kj = -i, \quad ik = -j.$$

How to remember these? Think of the cross product of basis vectors i, j, k, e.g. $i \times j$ etc.

Quaternions

- A quaternion $q = s + u_x i + u_y j + u_z k$ can be written in a scalar-vector notation as a pair $q = (s, u)$, where the vector $u = (u_x, u_y, u_z)^T$
- Let $q = (s, u)$, $p = (t, v)$ be quaternions and c a real number. Then
  - $q + p = (s, u) + (t, v) = s + u_x i + u_y j + u_z k + t + v_x i + v_y j + v_z k = (s + t, u + v)$.
  - $cq = (cs, cu)$. $(-1)q = -q = (-s, -u)$.
  - $qp = \cdots$ (use distributive law) $\cdots = (st - u \cdot v, sv + tu + u \times v)$. $c(qp) = (cq)p$.
  - Conjugation $q^* = s - u_x i - u_y j - u_z k = (s, -u)$. $cq^* = (cq)^*$.
  - Length $|q| = \sqrt{qq^*} = \sqrt{s^2 + u \cdot u}$. If $|q| = |p| = 1$, then $|qp| = 1$.
  - Additive unit quaternion (0,0), multiplicative unit quaternion (1,0).
  - $q^{-1} = \frac{1}{|q|^2} q^*$. If $|q| = 1$, then $q^{-1} = q^*$.
  - $(q + p)^* = q^* + p^*$.
  - $(qp)^* = p^* q^*$.
  - Dot product: $q \cdot p = st + u \cdot v$.
  - Addition and multiplication are associative. Only addition is commutative.

Rotation via quaternion

- Let $q = (s, u)$ be a quaternion s.t. $|q| = 1$. Then there exists an angle $\alpha \in (0, 2\pi)$ s.t. $q = (\cos\alpha, \sin\alpha\, v)$, where $v = 0$ if $|s| = 1$, else $v = u/\sin\alpha$.
  Proof:
  If $|s| = 1 \Rightarrow \alpha = 0$. Otherwise,
  $|q| = 1 \Rightarrow |s| < 1 \Rightarrow \alpha = \cos^{-1} s$ (choose α s.t. $\sin\alpha > 0$).
  $1^2 = |q|^2 = s^2 + u \cdot u = \cos^2\alpha + |u|^2 \Rightarrow |u|^2 = 1 - \cos^2\alpha = \sin^2\alpha \Rightarrow |u| = |\sin\alpha|$.
- Let $q = (s, u), p = (0, v)$ be quaternions s.t. $|q| = 1$. Then we define
  $$R_q(p) = qpq^{-1} = qpq^* = \cdots = (0, (s^2 - u \cdot u)v + 2(u \cdot v)u + 2s(u \times v)).$$
- Let us compare the Rodrigues' rotation formula with $R_q(p)$:
  $\cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v$ ← rotation formula
  $(s^2 - u \cdot u)v + 2(u \cdot v)u + 2s\, u \times v$ ← vector of $R_q(p)$
- Question: Is the vector part of $R_q(p)$ equal to Rodrigues' formula?

Rotation via quaternion

- $q = (s, u)$ can be expressed as $q = (\cos\alpha, \sin\alpha\, a)$, $s = \cos\alpha$, $a = u/\sin\alpha$
- Therefore, the vector part of $R_q(p)$ is:

$$(s^2 - u \cdot u)v + 2(u \cdot v)u + 2s\, u \times v =$$
$$(\cos^2\alpha - \sin^2\alpha)v + 2\sin^2\alpha\,(a \cdot v)a + 2\cos\alpha\sin\alpha\, a \times v.$$

So, the angle α must satisfy these three equalities (relevant trigonometry identities are on the right):

$\cos\theta = \cos^2\alpha - \sin^2\alpha$ // $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$1 - \cos\theta = 2\sin^2\alpha$ // $\cos 2\alpha = 1 - 2\sin^2\alpha$

$\sin\theta = 2\cos\alpha\sin\alpha$ // $\sin 2\alpha = 2\cos\alpha\sin\alpha$

A solution exists: $\theta = 2\alpha$. So, $R_q(p)$ rotates p about axis a by 2α.

- Observations: for quaternions q, q', p s.t. $|q| = |q'| = 1$:
  - $q^* R_q(p) q = q^*(qpq^*)q = p$ => q^* is the inverse rotation to q.
  - $R_{-q}(p) = (-q)p(-q)^* = qpq^* = R_q(p)$ => $-q$ is the same rotation as q.
  - $R_{q'}(R_q(p)) = R_{q'}(qpq^*) = q'(qpq^*)q'^* = (q'q)p(q'q)^* = R_{q'q}(p)$.

Quaternions and other representations

- Conversion axis-angle to quaternion: A vector $v \in R^3$ rotated about a unit axis $a \in R^3$ by an angle $\theta \in (0, 2\pi)$ can be computed using quaternions as $R_q(p) = qpq^*$, where $p = (0, v)$, $q = (\cos\theta/2, \sin\theta/2\, a)$.
- Conversion quaternion to axis-angle: Let q be a quaternion s.t. $|q| = 1$ expressed in the form $q = (\cos\alpha, \sin\alpha\, a)$, $|a| = 1$. Then q represents rotation about the axis vector a by the angle 2α.
- Conversion quaternion to rotation matrix: Let $q = (s, u)$ be a quaternion s.t. $|q| = 1$. Then the vector part of $R_q((0, v))$ is:

$$(s^2 - u \cdot u)v + 2(u \cdot v)u + 2s\, u \times v =$$

$$(s^2 - (1 - s^2))v + 2(u \times (u \times v) + (1 - s^2)v) + 2s\, u \times v =$$

$$v + 2[u]^2 v + 2s[u]v =$$

$$(I + 2[\![u]\!]^2 + 2s[\![u]\!])v$$

$|q|^2 = 1^2 = s^2 + u \cdot u$

$u \times (u \times v) = (u \cdot v)u - (u \cdot u)v$

$(u \cdot v)u = u \times (u \times v) + (1 - s^2)v$

Linear interpolation (lerp)

- Given quaternions q, p s.t. $|q| = |p| = 1$ we can linearly interpolate between them by $t \in (0,1)$:

$$Q(t) = \frac{(1-t)q + tp}{|(1-t)q + tp|}$$

- Technical issues related to $|(1-t)q + tp|$:
  - Slow – we must compute the square root.
  - The result is not defined when $=0$.

- Problem: The velocity is not constant (increases and decreases).
  $\Rightarrow$ Visible visual artefact – we prefer uniform blending between rotations
- Can we do better?
  - Yes: use spherical linear interpolation.

Spherical linear interpolation (slerp)

- Let us first compute a quaternion Δq representing a rotation from a quaternion $q_0 = (s, u)$ to $q_1 = (h, v)$. We assume $|\Delta q| = |q_0| = |q_1| = 1$.

  $\Rightarrow$ $\Delta q\, q_0 = q_1$

  $\Delta q = q_1 q_0^{-1} = q_1 q_0^* = (sh + u \cdot v, sv - hu + u \times v) = (\cos\alpha, \sin\alpha\, a)$.

  where,

  $\alpha = \cos^{-1}(sh + u \cdot v)$, $a = (sv - hu + u \times v)/\sin\alpha$ ($a = 0$ for $\alpha = 0, \pi$).

- For $t \in (0,1)$ we define $\Delta q(t) = (\cos t\alpha, \sin t\alpha\, a)$. So, we get

  $\mathrm{slerp}(q_0, q_1, t) = \Delta q(t) q_0 = (\cos t\alpha, \sin t\alpha\, a) q_0$.

Quaternion derivative

- Let $q(t) = s(t) + u_x(t)i + u_y(t)j + u_z(t)k = (s(t), \mathbf{u}(t))$, be a quaternion where $s(t), u_x(t), u_y(t), u_z(t)$ are functions of $t \in \mathbb{R}$. Then we define

$$\frac{dq(t)}{dt} = \dot{q}(t) = \lim_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} + i \lim_{\Delta t \to 0} \frac{u_x(t+\Delta t) - u_x(t)}{\Delta t} + j \lim_{\Delta t \to 0} \frac{u_y(t+\Delta t) - u_y(t)}{\Delta t} + k \lim_{\Delta t \to 0} \frac{u_z(t+\Delta t) - u_z(t)}{\Delta t} =$$

$$= \frac{ds(t)}{dt} + \frac{du_x(t)}{dt} i + \frac{du_y(t)}{dt} j + \frac{du_z(t)}{dt} k =$$

$$= \left(\frac{ds(t)}{dt}, \frac{d\mathbf{u}(t)}{dt} \right).$$

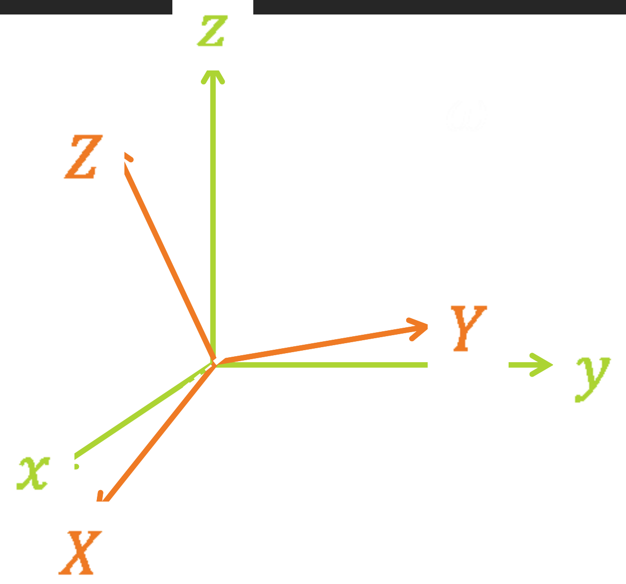
- Note: $\int q(t)dt = \int s(t)dt + i \int u_x(t)dt + j \int u_y(t)dt + k \int u_z(t)dt.$

Quaternion derivative

- Example: Let's compute a derivative of the orientation of a frame of reference (orange) rotating a constant angular speed $|\omega|$ about the unit axis vector $\bar{\omega} = \omega/|\omega|$.
- Solution:
  - Let $q(t)$ be an orientation (rotation) of the orange frame in the green one (world)
  - We express the rotation about $\bar{\omega}$:

$$\Delta q(\Delta t) = \left(\cos\frac{|\omega|\Delta t}{2}, \sin\frac{|\omega|\Delta t}{2}\bar{\omega}\right).$$

  - $\dot{q}(t) = \lim_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta q(\Delta t) q(t) - q(t)}{\Delta t} = \left(\lim_{\Delta t \to 0} \frac{\Delta q(\Delta t) - (1, 0)}{\Delta t}\right) q(t)$



Quaternion derivative

Quaternion representation

- We can use quaternions to express any orientation of object in 3D space.

```
public class Orientation {   // Equals to a quaternion q=(s,u), |q|=1.
    float s;                 // the scalar part
    Vector3 u;               // the vector part
};
```

- Pros:
  - Low memory footprint.
  - Fast conversion to rotation matrix (no need to compute cosine & sine).
  - Fast composition of rotations (just multiply the quaternions).
  - We can use lerp and slerp.
- Cons:
  - Less human readable.

What representation to use?

- There is no single winner – each representation has pros and cons.

- Examples:
  - When specifying a rotation along a **coordinate** axis (e.g., world Z), then Euler angles are a good choice.
  - When specifying a rotation along **non-coordinate** axis, then axis-angle representation is a good choice.
  - When composing rotations of some joint of a skeleton, then quaternions can do it quickly.
  - When rotations must be composed with other transformations, then use matrix representation.

- Game engine should provide **all** representations and conversions between them.

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