Collision detection

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PA199

Outline

- Broad phase
  - Sweep and prune algorithm
- Narrow phase
  - Gilbert-Johnson-Keerthi (GJK) algorithm
- Caching collisions
- Computing collision time

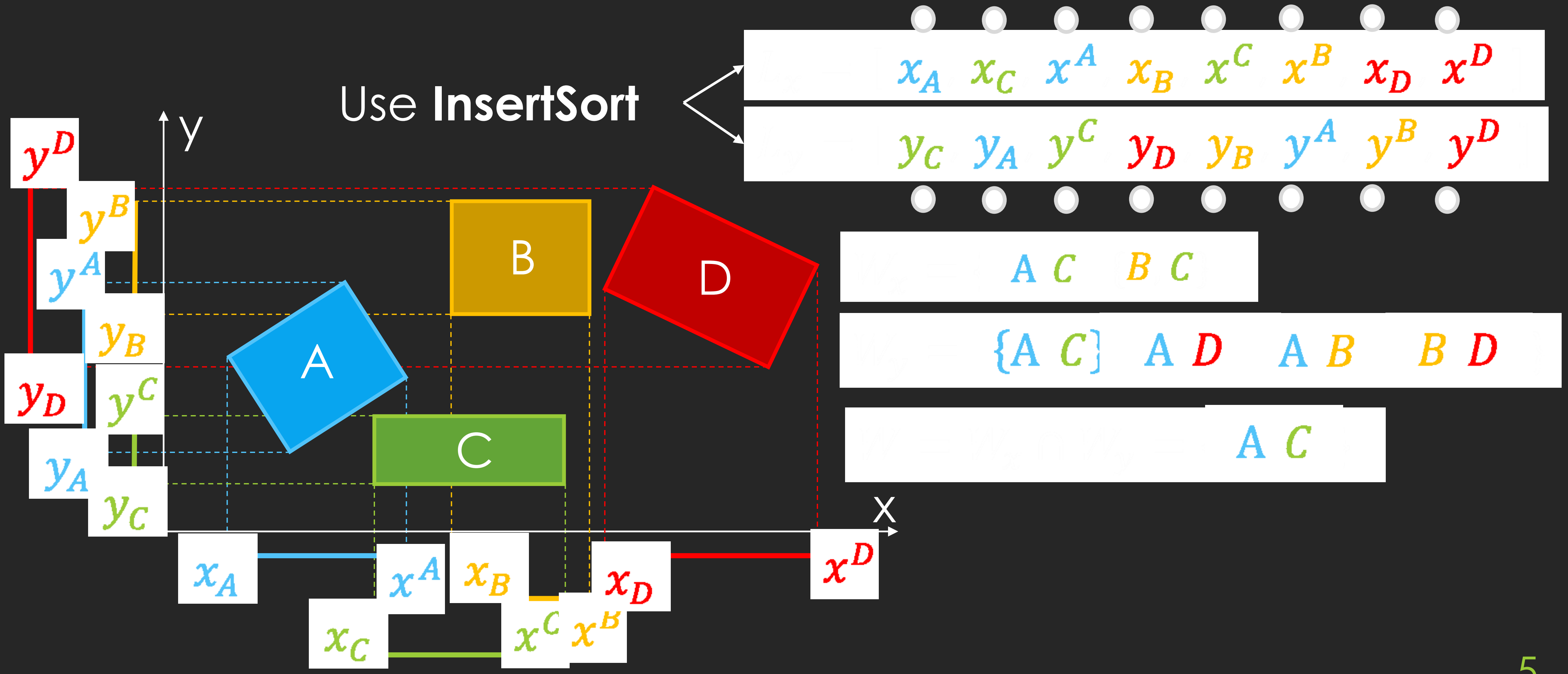
Broad phase

Broad phase

- The goal is to quickly find **pairs** of **potentially colliding** rigid bodies.
  - Used algorithm defines meaning of “potentially colliding”. Examples:
    - When AABBs of the bodies are colliding.
    - When both bodies are in the same area of space.
- We can use space partitioning data structures we already know:
  - Octree, k-D tree, BSP
- Rigid bodies change their positions and orientations during simulation.

  => The data structure must be periodically updated.
  - Utilize time coherence of frames (positions of bodies do not change much between adjacent frames) to get an efficient update algorithm.

Sweep and prune algorithm



Sweep and prune algorithm

- The presented version is easy to understand and implement.
- But it wastes time by recomputing W from scratch each timestep.
- In practice, we use an improved version:
  - We start with the arrays L_α, $\alpha \in \{x, y, z\}$, and W from the previous frame.
  - We incrementally update each L_α and W for each relocated object A.

```
foreach axis α ∈ {x, y, z} do
    Update a_α in L_α by the new lower bound of A along the axis α.
    while ∃ a_α^X right before a_α in L_α s.t. a_α^X > a_α do
        Swap a_α with a_α^X in L_α.
        if X is None then insert (A, X) to W.
```

Sweep and prune algorithm

- Update α^A in L_α by the new upper bound of A along the axis α.

while $\exists \alpha_l^X$ right after α^A in L_α s.t. $\alpha^A > \alpha_l^X$ **do**

  Swap α^A with α_l^X in L_α.

  if Y **is None then** Insert (A, X) to W.

while $\exists \alpha_l^X$ right after α_A in L_α s.t. $\alpha_A > \alpha_l^X$ **do**

  Swap α_A with α_l^X in L_α.

  if X **is None then** Erase (A, X) from W.

while $\exists \alpha_l^X$ right before α^A in L_α s.t. $\alpha_l^X > \alpha^A$ **do**

  Swap α^A with α_l^X in L_α.

  if Y **is None then** Erase (A, X) from W.

Moving α^A "to the right"

Moving α^A "to the left"

Sweep and prune algorithm

- Possible memory representation of the lists L_a, $a \in \{x, y, z\}$

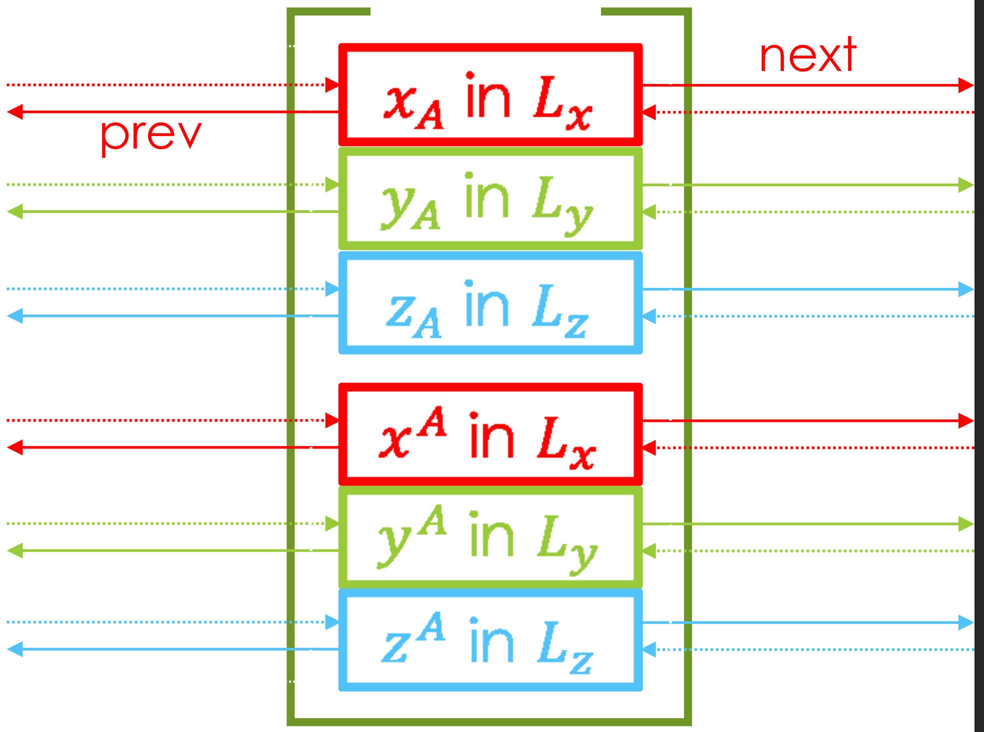
```
struct Link {
    Link *next, *prev;
    float coord;
    char lohi : 1;
};
```

If lohi == 0, then "coord" is a_A and a^A otherwise.

"red"

Represents either a_A or a^A.

```
using AABB = Link[2][3];
```



Sweep and prune algorithm

- If "p" is a pointer to a "Link" of the list $L_\alpha, \alpha \in \{0,1,2\}$ (i.e., $\{x, y, z\}$), then we can convert it to a pointer to AABB using the pointer arithmetic:

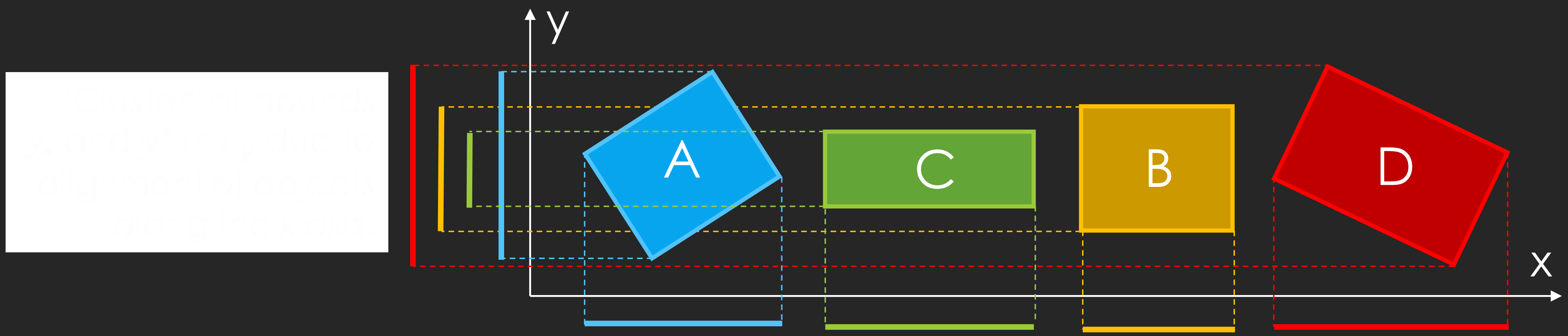
  (AABB*)(p − (α + 3 * (int)p.lohi) * sizeof(Link))

- Represent the set W as a dictionary of pairs of object IDs.
  - Sort the pairs s.t. the lower ID comes first and the other the second.

- Initialize the data structure to contain a single auxiliary AABB s.t.:
  - Values in the links are: $x_A = y_A = z_A = -\infty$ and $x^A = y^A = z^A = +\infty$.
  - All 2*3 links are properly interconnected in the lists L_x, L_y, L_z.
  - This auxiliary AABB avoids the "nullptr" checks in the algorithm (loops).

Sweep and prune algorithm

- Performance of the algorithm is sensitive to alignment of objects along coordinate axes:

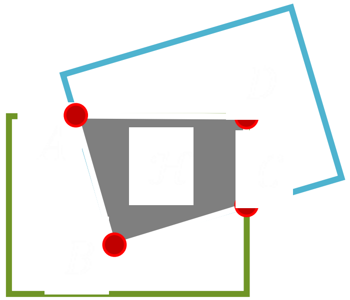


- A relocation of an object leads to a lot of swaps thought the "cluster" in the array.

Narrow phase

Narrow phase

- The goal is for each pair of potentially colliding shapes to
  - Decide whether the shapes really collide or not
  - Compute a finite model of the (infinite) set of collision points
- Example: Find finite and minimal number of points in $\mathcal{X}$ whose convex hull contains $\mathcal{X}$



- Requirement: The effect of collision forces computed at points of the model must be equal to collision forces computed at all points in $\mathcal{X}$

Gilbert-Johnson-Keerthi (GJK) algorithm

- Decides whether two **convex** shapes have **empty intersection** or not.

Convex shapes   Concave shapes

  - We can approximate a concave shape by a **set** of convex shapes.
  - For the empty intersection we can obtain a pair of the **closest points**.

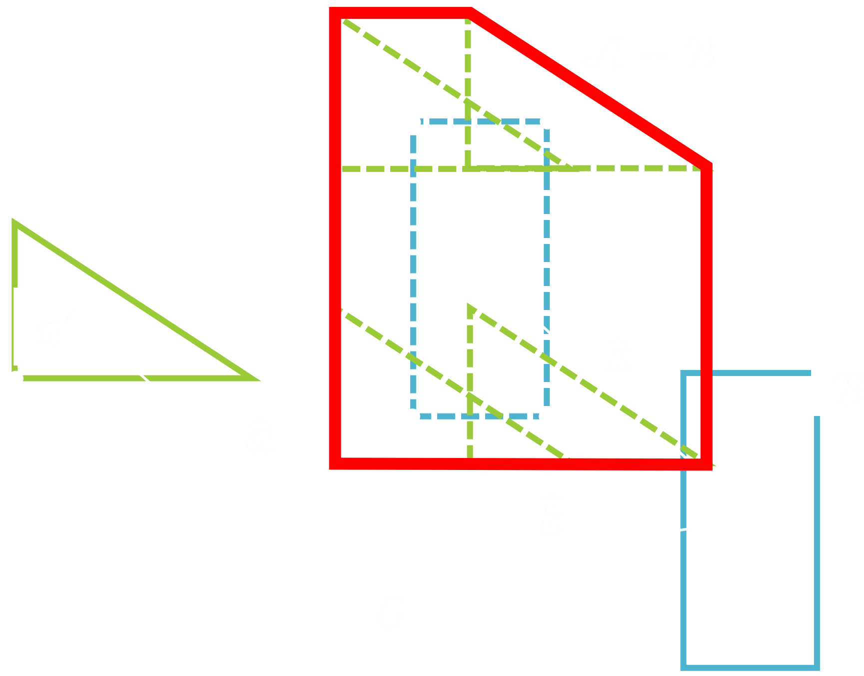
- We must first build a terminology:
  - Minkowski sum and difference
  - Simplex
  - Support function

GJK: Minkowski sum

- Minkowski sum: $\mathcal{A} + \mathcal{B} = \{a + b; a \in \mathcal{A} \wedge b \in \mathcal{B}\}$.
- How to draw Minkowski sum?
  - Choose some points $\hat{a} \in \mathcal{A} \wedge \hat{b} \in \mathcal{B}$.
  - Then, $\forall a \in \mathcal{A}\ \exists a'$ s.t. $a = \hat{a} + a'$.
  - Therefore, for each $a \in \mathcal{A} \wedge b \in \mathcal{B}$

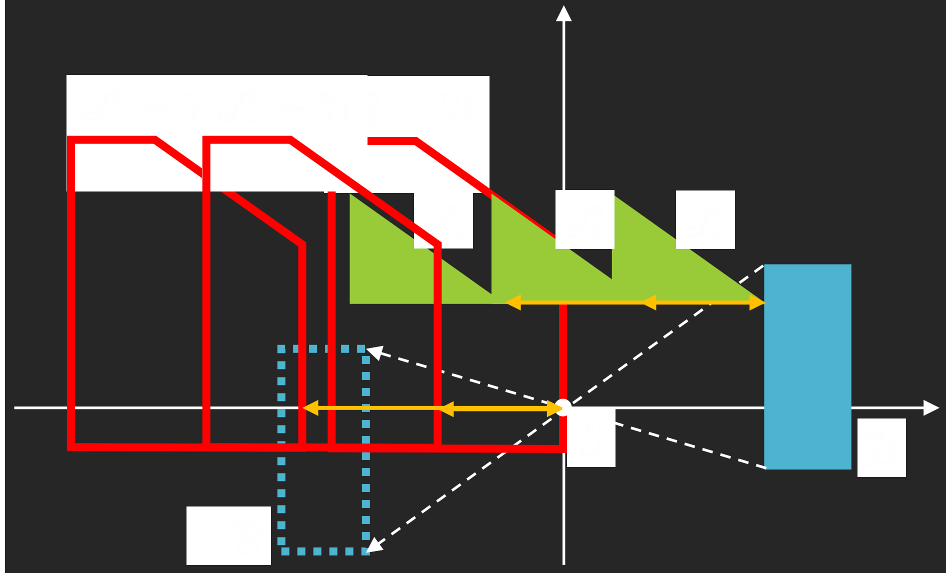
$$a + b = \hat{a} + \hat{b} + a' + b'$$

  - So, we draw $\mathcal{A} + \mathcal{B}$ around $\hat{a} + \hat{b}$.
    - Draw $\mathcal{B}$ around $\hat{a} + \hat{b}$.
    - Draw $\mathcal{A}$ around $\mathcal{B}$'s perimeter.
    - $\mathcal{A} + \mathcal{B}$ is the convex hull.



GJK: Minkowski difference

- **Minkowski difference:** $\mathcal{A} - \mathcal{B} = \mathcal{A} + (-\mathcal{B})$, where $-\mathcal{B} = \{-\boldsymbol{b}; \boldsymbol{b} \in \mathcal{B}\}$
- **Lemma:** The shortest distance between $\mathcal{A}$ and $\mathcal{B}$ is equal to the distance of $\mathcal{A} - \mathcal{B}$ to the origin.
  Proof: It is a length of the shortest $\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}}$, s.t. $\hat{\boldsymbol{a}} \in \mathcal{A} \wedge \hat{\boldsymbol{b}} \in \mathcal{B}$. But $\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}} \in \mathcal{A} - \mathcal{B}$.
- **Consequence:** Shapes $\mathcal{A}$ and $\mathcal{B}$ collide if and only if $\mathcal{A} - \mathcal{B}$ contains the origin.

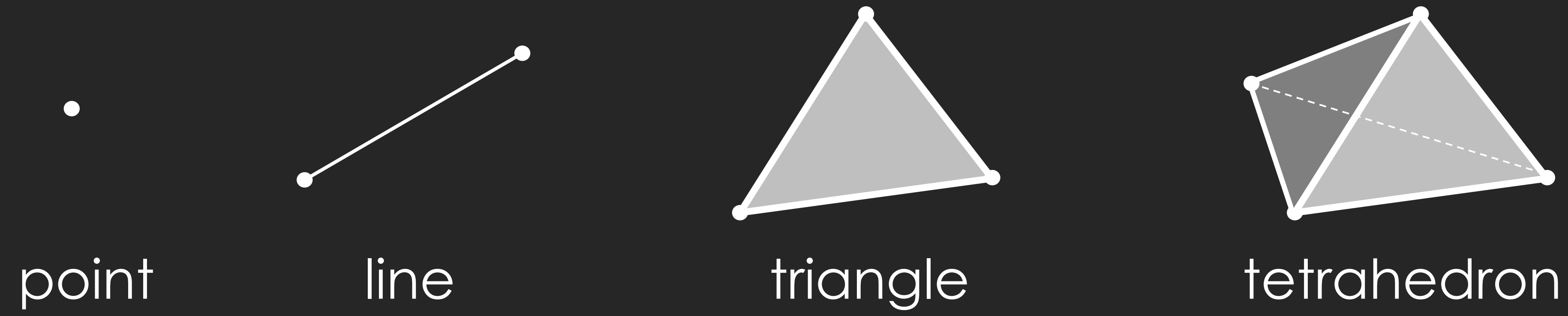


GJK: Minkowski difference

- Lemma: If shapes $\mathcal{A}$ and $\mathcal{B}$ are convex, then $\mathcal{A} - \mathcal{B}$ is also convex.
  Proof: For each $u, v \in \mathcal{A} - \mathcal{B}$ there exist $a_u, a_v \in \mathcal{A}$ and $b_u, b_v \in \mathcal{B}$ s.t.
  $u = a_u - b_u$ and $v = a_v - b_v$. Then, for $t \in (0,1)$, we get
  $$u + t(v - u) = (a_u - b_u) + t\big((a_v - b_v) - (a_u - b_u)\big) =$$
  $$a_u - b_u + ta_v - tb_v - ta_u + tb_u =$$
  $$a_u + t(a_v - a_u) - \big(b_u + t(b_v - b_u)\big).$$
  $\mathcal{A}$ and $\mathcal{B}$ are convex => $a_u + t(a_v - a_u) \in \mathcal{A}$, $b_u + t(b_v - b_u) \in \mathcal{B}$ =>
  $a_u + t(a_v - a_u) - (b_u + t(b_v - b_u)) \in \mathcal{A} - \mathcal{B}$ => $\mathcal{A} - \mathcal{B}$ is convex.
- Consequence: If the origin lies in the convex hull of points $a_1, \ldots, a_n \in \mathcal{A} - \mathcal{B}$, then convex shapes $\mathcal{A}$ and $\mathcal{B}$ have non-empty intersection.

GJK: Simplex

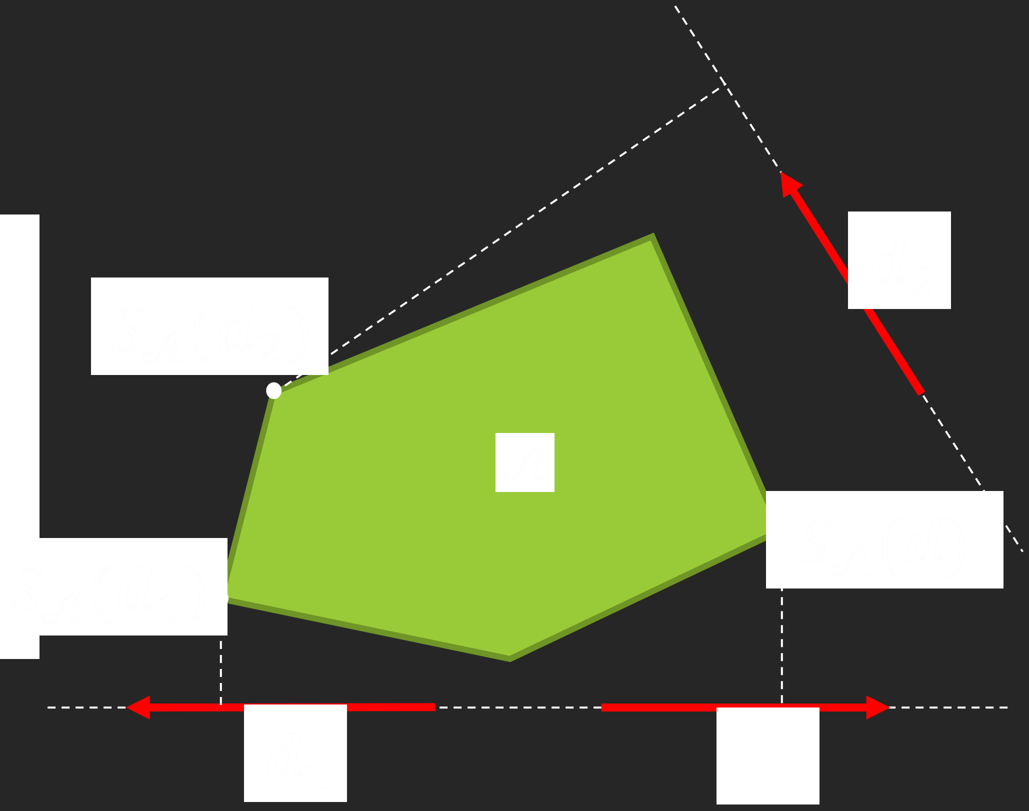
- A **simplex** is a convex hull of an affinely independent points.



- GJK searches for a simplex s.t. origin lies inside or prove that no such simplex exists.
- Note: In 2D case we only need point, line and triangle.

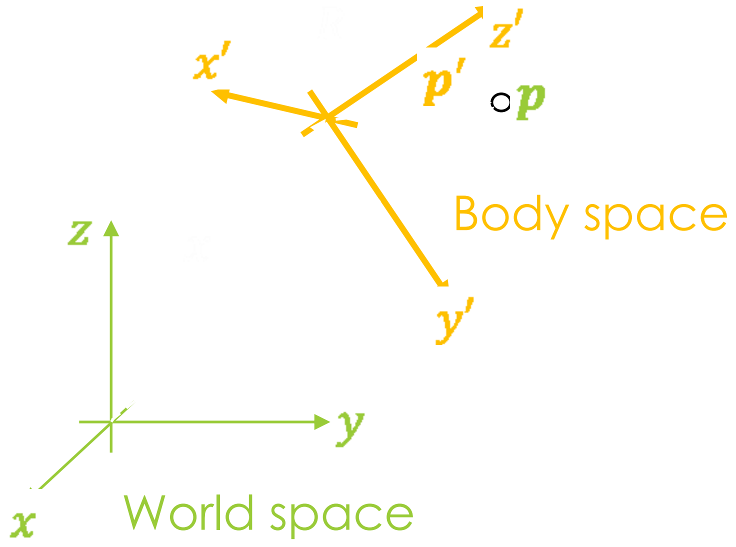
GJK: Support function

- Given a shape A and a non-zero vector d, a support function S_A returns a point $S_A(d) \in A$ s.t. $S_A(d) \cdot d = \max\{x \cdot d : x \in A\}$.



GJK: Support function

- A shape $\mathcal{A}$ can be defined in a local system – body/model space.



$$\boldsymbol{p} = R\boldsymbol{p}' + x$$

$$\boldsymbol{p}' = R^{-1}(\boldsymbol{p} - x)$$

- Therefore, this must be reflected in the computation of $S_{\mathcal{A}}(\boldsymbol{d})$.

GJK: Support function

- When a convex shape $\mathcal{A}$ is defined in body space (R, x), then we denote $R\mathcal{A} + x$ the corresponding convex shape in the world space.
  - More precisely: $R\mathcal{A} + x = \{Rp' + x; p' \in \mathcal{A}\}$.
- Lemma: $S_{R\mathcal{A}+x}(\vec{d}) = RS_{\mathcal{A}}(R^T\vec{d}) + x$, for each world-space vector $\vec{d} \neq \vec{0}$.

Proof: First, we show that $\forall p' \in \mathcal{A}$ the following equality (*) holds true

$$\begin{aligned}(Rp' + x) \cdot \vec{d} &= (Rp') \cdot \vec{d} + x \cdot \vec{d} \\ &= \vec{d}^T(Rp') + x \cdot \vec{d} \\ &= (\vec{d}^T R)p' + x \cdot \vec{d} \\ &= (R^T\vec{d})^T p' + x \cdot \vec{d} \\ &= p' \cdot (R^T\vec{d}) + x \cdot \vec{d}\end{aligned}$$

GJK: Support function

- Now, $S_{RA+x}(\vec{d}) \cdot \vec{d} = \max\{p \cdot \vec{d},\ p \in RA + x\}$

  $= \max\{(Rp' + x) \cdot \vec{d},\ p' \in A\}$

  $= \max\{p' \cdot (R^T \vec{d}) + x \cdot \vec{d},\ p' \in A\}$ according to (*)

  $= \max\{p' \cdot (R^T \vec{d}),\ p' \in A\} + x \cdot \vec{d}$

  $= S_A(R^T \vec{d}) \cdot R^T \vec{d} + x \cdot \vec{d}$

  $= (RS_A(R^T \vec{d}) + x) \cdot \vec{d}$ according to (*)

  Therefore, $S_{RA+x}(\vec{d}) = RS_A(R^T \vec{d}) + x$.

GJK: Support function examples

- $\mathcal{A}$ is a sphere at the origin with the radius r:

$$S_{\mathcal{A}}(\vec{d}) = r\frac{\vec{d}}{|\vec{d}|}$$

- $\mathcal{A}$ is an axis aligned bounding box (AABB) at the origin with sizes $2s_x, 2s_y, 2s_z$ along corresponding coordinate axes:

$$S_{\mathcal{A}}(\vec{d}) = \left(\operatorname{sgn}(d_x)\, s_x, \operatorname{sgn}(d_y)\, s_y, \operatorname{sgn}(d_z)\, s_z\right)^T$$

where $\operatorname{sgn}(\alpha) = \begin{cases} -1 & \text{if } \alpha < 0 \\ 1 & \text{otherwise} \end{cases}$

GJK: Support function examples

- $\mathcal{A}$ is a cylinder at the origin with the central axis aligned with the z coordinate axis, with the radius r and with the top and bottom base at z-coordinate h and $-h$, respectively:

$$S_{\mathcal{A}}(\vec{d}) = \begin{cases} \left(\frac{r}{\sigma} d_x, \frac{r}{\sigma} d_y, \operatorname{sgn}(d_z)\, h\right)^T & \text{if } \sigma > 0 \\ \left(0, 0, \operatorname{sgn}(d_z)\, h\right)^T & \text{otherwise} \end{cases}$$

  where $\sigma = \sqrt{d_x^2 + d_y^2}$, and $\operatorname{sgn}(x)$ was defined earlier.

- $\mathcal{A}$ is any convex polytope (e.g., point, line, triangle, convex polygon, tetrahedron, box, …) with vertices $V = \{v_1, \dots, v_n\}$:

$$S_{\mathcal{A}}(\vec{d}) = v_k \quad \text{s.t.} \quad v_k \cdot \vec{d} = \max\{v_i \cdot \vec{d} : v_i \in V\}$$

GJK: Support function

- Lemma: $S_{\mathcal{A}-\mathcal{B}}(\vec{d}) = S_{\mathcal{A}}(\vec{d}) - S_{\mathcal{B}}(-\vec{d})$.

  Proof: $S_{\mathcal{A}-\mathcal{B}}(\vec{d}) \cdot \vec{d} = \max\{(a-b)\cdot \vec{d};\, a \in \mathcal{A} \wedge b \in \mathcal{B}\}$

  $$= \max\{a\cdot \vec{d};\, a \in \mathcal{A}\} - \min\{b \cdot \vec{d};\, b \in \mathcal{B}\}$$

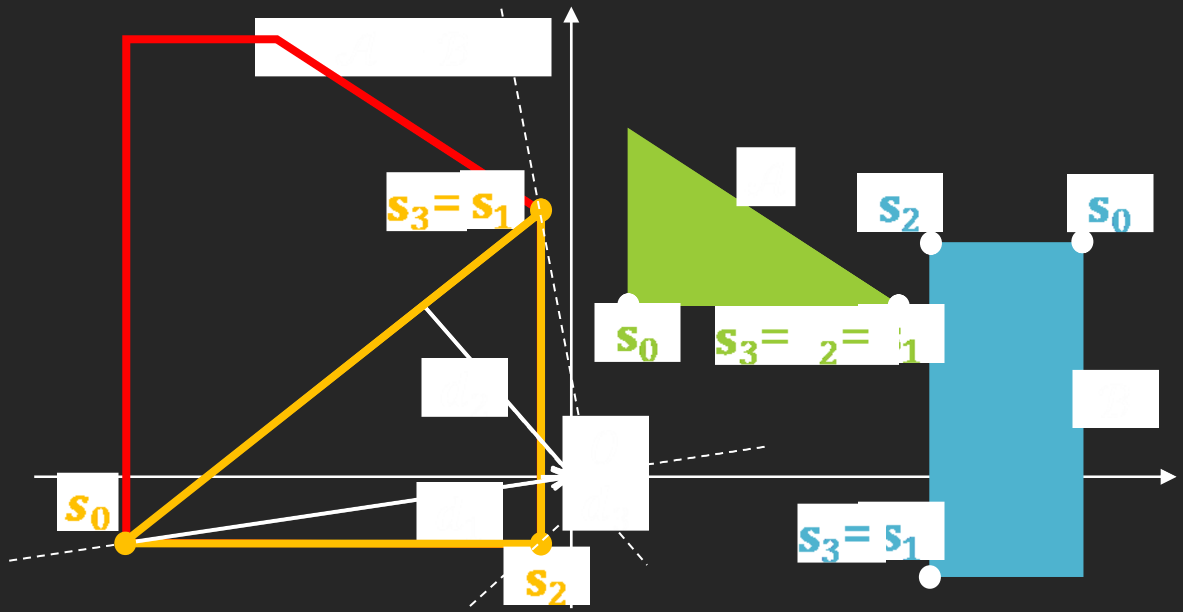
  $$= S_{\mathcal{A}}(\vec{d})\cdot \vec{d} + \max\{b\cdot(-\vec{d});\, b \in \mathcal{B}\}$$

  $$= S_{\mathcal{A}}(\vec{d})\cdot \vec{d} + S_{\mathcal{B}}(-\vec{d})\cdot(-\vec{d})$$

  $$= (S_{\mathcal{A}}(\vec{d}) - S_{\mathcal{B}}(-\vec{d}))\cdot \vec{d}$$

- We therefore do not have to construct $\mathcal{A}-\mathcal{B}$ and $S_{\mathcal{A}-\mathcal{B}}$. We work with the given shapes $\mathcal{A}$ and $\mathcal{B}$ and their support functions.

GJK: The algorithm – intuition (2D case)



s_0 s_0

$S = \{s_0, s_1, s_2$

s_0 s_0 s_0

s_1 s_1 s_1

s_1

s_2 s_2 s_2

s_2

s_3 s_3 s_3

s_3 $<$ NO INTERSECTION!

GJK: The algorithm – intuition (2D case)

- Since A and B have empty intersection, we can compute a pair of closest points:
  - First, we find the closest point X of the simplex $S = \{\mathbf{s}_1, \mathbf{s}_2\}$ to the origin, that is $X = \mathbf{s}_1 + t(\mathbf{s}_2 - \mathbf{s}_1)$ for some $t \in (0,1)$ s.t. $\|X - 0\|^2 = \|\mathbf{s}_1 + t(\mathbf{s}_2 - \mathbf{s}_1)\|^2$ is minimal. So, solve the equation:

$$\frac{d}{dt}\|\mathbf{s}_1 + t(\mathbf{s}_2 - \mathbf{s}_1)\|^2 = 0$$

$$\frac{d}{dt}\left(\mathbf{s}_1 \cdot \mathbf{s}_1 + 2t\,\mathbf{s}_1 \cdot (\mathbf{s}_2 - \mathbf{s}_1) + t^2(\mathbf{s}_2 - \mathbf{s}_1)^2\right) = 0$$

$$2\,\mathbf{s}_1 \cdot (\mathbf{s}_2 - \mathbf{s}_1) + 2t(\mathbf{s}_2 - \mathbf{s}_1)^2 = 0$$

$$t = -\frac{\mathbf{s}_1 \cdot (\mathbf{s}_2 - \mathbf{s}_1)}{(\mathbf{s}_2 - \mathbf{s}_1)^2}$$

← Also clip to (0,1)

GJK: The algorithm – intuition (2D case)

- Then, find the corresponding points in the shapes $\mathcal{A}$ and $\mathcal{B}$

$$s_1 + t(s_2 - s_1) = (s_1 - s_1) + t((s_2 - s_2) - (s_1 - s_1))$$

$$= \underbrace{s_1 + t(s_2 - s_1)}_{\in \mathcal{A}} - \underbrace{(s_1 + t(s_2 - s_1))}$$

GJK: The algorithm

▶ Choose some $p \in \mathcal{A} - \mathcal{B}$. // Usually, $\boldsymbol{p}$ comes from the previous frame.

$S = \emptyset$ // We start with the empty simplex.

$s = S_{\mathcal{A}-\mathcal{B}}(-p)$ // NOTE: Our direction vector $\boldsymbol{d}$ to the origin is just $-\boldsymbol{p}$.

while $|p|^2 - p \cdot s > \epsilon^2$ **do** // Proving termination condition: see [4].

// $\boldsymbol{p}$ is still far from the closest point of $\mathcal{A} - \mathcal{B}$ to the origin.

$p =$ closest_to_origin(convex_hull($S \cup \{s\}$))

Point, line, triangle, or tetrahedron.

Can be computed quickly for shapes

$S =$ smallest $X \subseteq S \cup \{s\}$ s.t. $p \in$ convex_hull(X) // Reduce the simplex.

$s = S_{\mathcal{A}-\mathcal{B}}(-p)$

return $a \in \mathcal{A}, b \in \mathcal{B}$ s.t. $p = a - b$ // $|\boldsymbol{p}|$ is the closest distance.

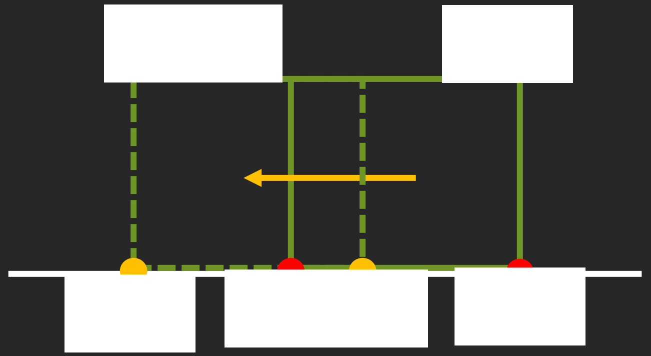
Caching collisions

Caching collisions

- Efficiency of the PGS algorithm for a constraint system depends on the initial value λ^0.
- It's likely that λ computed for a collision constraint at current frame would be "almost valid" for the next frame (if the collision persists).
- Therefore, caching λ values for collision (and other types of) constraints amongst frames can bring considerable speed boost.
- How to match collisions computed in different frames?

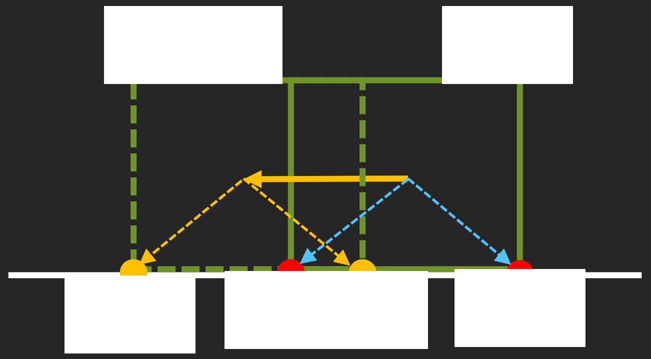
Caching collisions

- There are several possibilities:
  - **Distance** between collision points in **world space**:

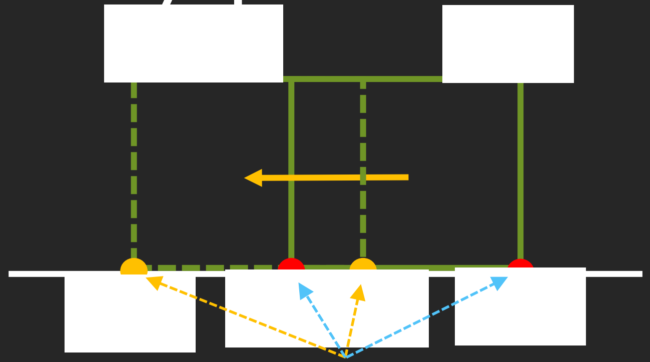


Imprecise.

  - **Distance** between collision points in **body space**:



Precise in space of the **box**



Imprecise in space of **ground**

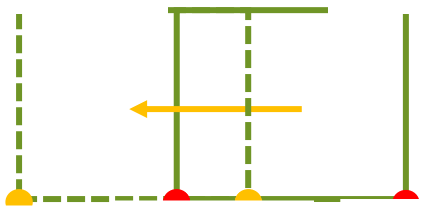
Caching collisions

- Identify collisons by geometrical properties of collision shapes:

```
enum GTYPE { VERTEX, EDGE, FACE };
struct CollisionID {
    int body_index_1;           // The index of R_i: i
    GTYPE feature_type_1;       // The type of colliding geometry in R_i
    int feature_index_1;        // Index of the colliding geometry in R_i
    int body_index_2;           // The index of R_j: j
    GTYPE feature_type_2;       // The type of colliding geometry in R_i
    int feature_index_2;        // Index of the colliding geometry in R_i
};
```

Define also comparison and hashing of CollisionID instances.

Caching collisions



We got precise mapping:

$id(a) - id(c) \rightarrow a > c$

$id(b) - id(d) \rightarrow b > d$

Recommended approach

- The cache should be a map from CollisionID instance to values λ:

```
using collision_cache = std::unordered_map<CollisionID,float>;
```

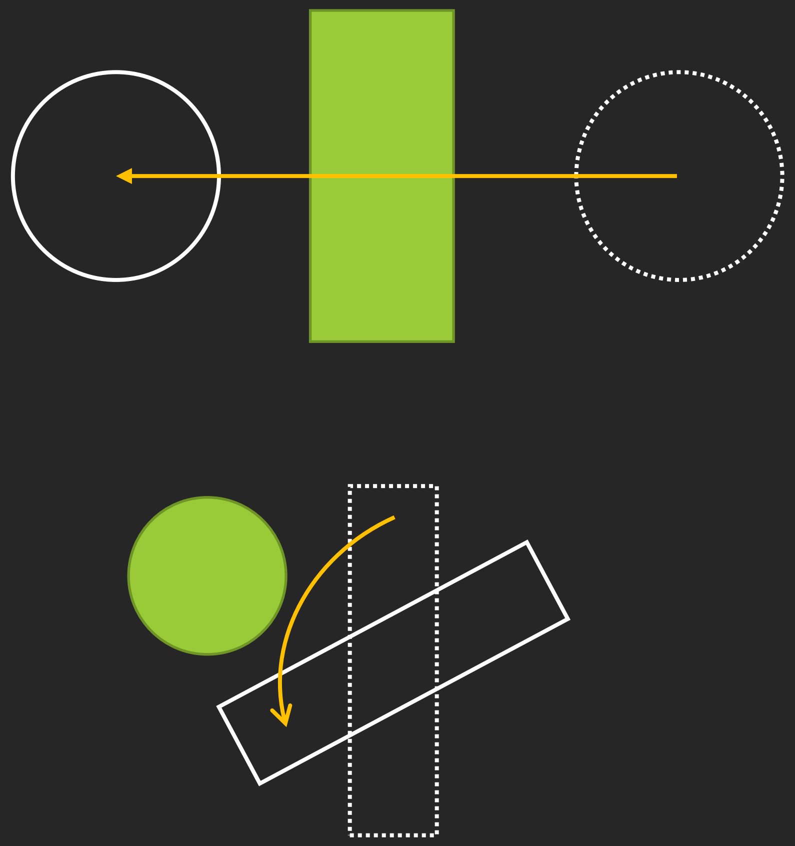
- And how to use the cache?

Caching collisions

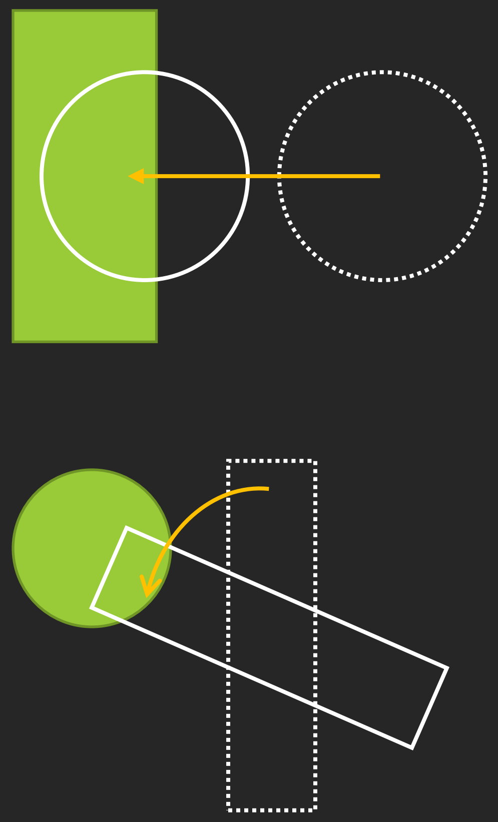
- Before solving the constraint system initialize λ^0 s.t.:
  - For each computed collision c and the corresponding element λ_i^0:
    - Build the Collision Instance id from c.
    - If id is present in the cache, then set λ_i^0 to the value λ in the cache.
    - Otherwise, set λ_i^0 to 0.
- Once new solution λ is computed updated the cache as follows:
  - Clear the cache.
  - For each collision c and the corresponding computed value λ:
    - Build the Collision Instance id from c.
    - Insert the mapping $id \to \lambda$ to the cache.

Computing collision time

Tunnelling and penetration



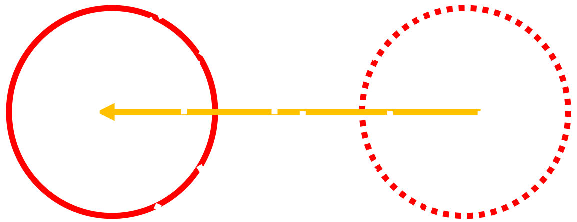
Tunnelling

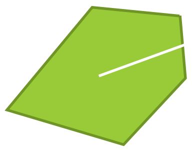


Penetration

Dealing with tunnelling and penetration

- The simplest approach is to subdivide the game time step dt into several small "internal" time steps.
- For broad phase:
  - Approximate collision shapes of bodies by "moving spheres":

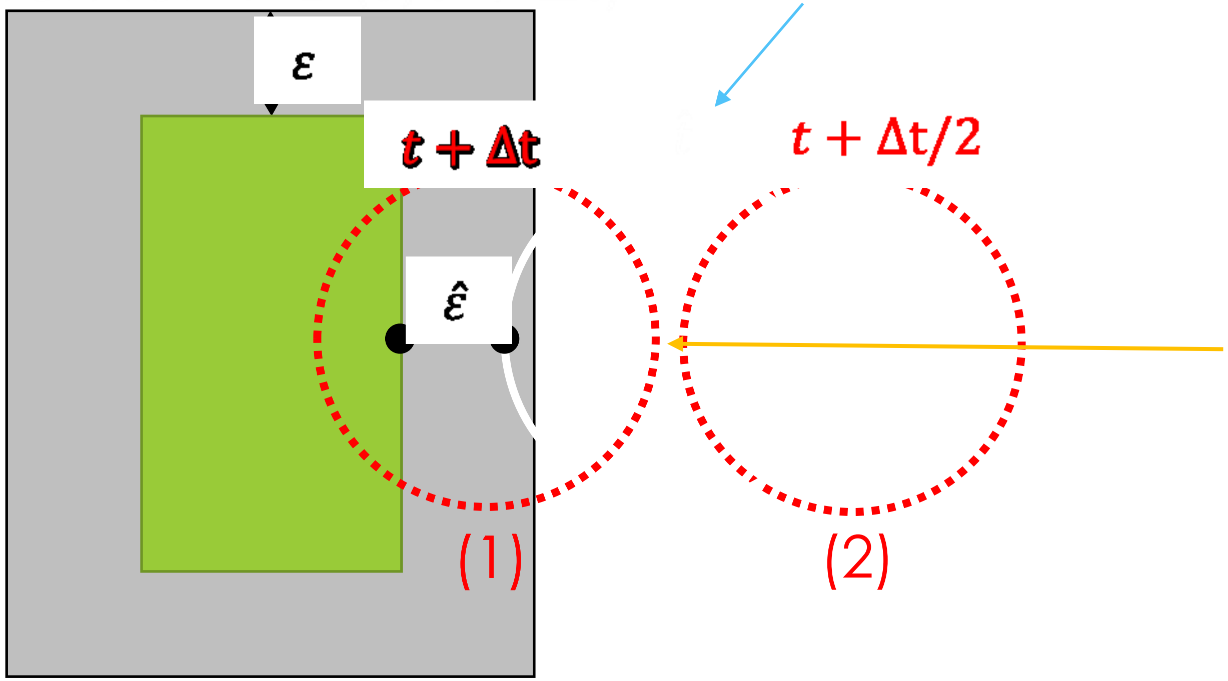




- Use the adaptive time step:
  - For each pair of potentially colliding shapes compute the nearest collision time.
  - Move the bodies only to the minimum of all nearest collision times.

Computing collision time

- Binary search for $\hat{t} \in (t, t + \Delta t)$



- There are 4D collision algorithms – they consider translations and rotations of tested objects.

References

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[2] E. G. Gilbert, D. W. Johnson and S. S. Keerthi; A fast procedure for computing the distance between complex objects in three-dimensional space; Journal on Robotics and Automation, vol. 4, no. 2, pp. 193-203, April 1988

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