Vícehodnotové logiky

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Outline

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Classical (two-valued) logic

Motivation:

Propositions are either true or false

Truth tables:

$\neg$	
0	1
1	0

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	1	1
1	0	1

Validates:

$A \vee \neg A$, $\neg(A \wedge \neg A)$, $A = \neg\neg A$, $A, \neg A \models B$, ...

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Multiple-valued logics

Motivation:

- Future contingents, undefined values, error states
- Truth-value gaps, gluts, degrees
- Mathematical generalization of classical logic

History:

Chrysippus, Peirce (unpublished), Łukasiewicz (1913), Post, Kleene, Tarski, . . . Goguen, . . . , Pavelka, Novák, Hájek, . . .

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Weak Kleene (aka Bochvar) three-valued logic

Motivation:

Undetermined/undefined truth values

Truth tables:

¬	
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X	X
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∧	0	X	1
0	0	X	0
X	X	X	X
1	0	X	1

∨	0	X	1
0	0	X	1
X	X	X	X
1	1	X	1

→	0	X	1
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No tautologies, but a distinct consequence relation (preserving the value 1 = the *designated* truth value)

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Strong Kleene three-valued logic

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Undetermined/undefined truth values, but $0 \wedge A = 0$ etc.

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McCarthy three-valued logic

Motivation:

Undetermined truth value, sequential evaluation

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No tautologies, non-commutative ∧,∨

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Łukasiewicz three-valued logic

Motivation:

Future contingents (failed), possibility (failed), half-truth

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Does have tautologies (cf. $A \rightarrow A$)

Invalidates: $A \vee \neg A$. Validates: $\neg\neg A = A$

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Gödel three-valued logic

Motivation:

Will be apparent later (a 3-valued instance of general Gödel logics)

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Post three-valued logic

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$A \wedge B \equiv_{\mathrm{df}} \neg(\neg A \vee \neg B)$

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Priest's Logic of Paradox

Motivation:

Paradoxical propositions (liar etc), dialetheism (true contradictions)

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But: both 0 and X are designated in LP

The same tautologies as in classical logic (incl. $A \vee \neg A$, $\neg(A \wedge \neg A)$), but a different consequence relation: eg, $A, \neg A \not\models B$ (paraconsistency)

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Dunn–Belnap's four-valued logic

Motivation:

Accommodate both underdetermined and conflicting data
Caution: not epistemic states (truth-functionality)

Truth tables:

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0	1
B	B
N	N
1	0

$\wedge$	0	B	N	1
0	0	0	0	0
B	0	B	0	B
N	0	0	N	N
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$1 = \{t\}, 0 = \{f\}, B = \{t, f\}, N = \emptyset$
Bilattice (information/truth order)
No good implication connective known

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Features of three-valued logics

Classical logic = restriction to $\{0, 1\}$, many possible connectives

Truth functionality = algebraic matrix semantics

Additional connectives:

$\triangle$	
0	0
X	0
1	1

$\otimes$	0	X	1
0	0	0	0
X	0	0	X
1	0	X	1

$\oplus$	0	X	1
0	0	X	1
X	X	1	1
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...

Łukasiewicz logic invalidates $A \wedge (A \to B) \to B$ and $A \vee \neg A$
but does validate $A \otimes (A \to B) \to B$ and $A \oplus \neg A$

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Lukasiewicz finitely-valued logics

Truth values:

$\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$

Truth tables:

$$\neg A = 1 - A$$
$$A \wedge B = \min(A, B)$$
$$A \vee B = \max(A, B)$$
$$A \rightarrow B = \max(1 - A + B, 1)$$
$$A \otimes B = \max(A + B - 1, 0)$$
$$A \oplus B = \min(A + B, 1)$$

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Post finitely-valued logics

Truth values:

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Truth tables:

$$\neg A = A - \frac{1}{n-1} \text{ if } A \neq 0, \text{ otherwise } 1$$
$$A \vee B = \max(A, B)$$
$$A \wedge B = \neg(\neg A \vee \neg B)$$

Invalidates: $\underbrace{\neg \ldots \neg}_{n-1} A = A$

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Gödel finitely-valued logics

Truth values:

$\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$

Truth tables:

$$\neg A = 1 \text{ if } A = 0, \text{ otherwise } 0$$
$$A \wedge B = \min(A, B)$$
$$A \vee B = \max(A, B)$$
$$A \rightarrow B = 1 \text{ if } A \leq B, \text{ otherwise } B$$

Gödel finitely-valued logics

Truth values:

$\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, 1\}$

Truth tables:

$$
\begin{aligned}
\neg A &= 1 \text{ if } A = 0, \text{ otherwise } 0 \\
A \wedge B &= \min(A, B) \\
A \vee B &= \max(A, B) \\
A \rightarrow B &= 1 \text{ if } A \leq B, \text{ otherwise } B
\end{aligned}
$$

Outline

Frequently employed principles

- Lattice or linear order of truth values
- Implication internalizes order: $A \to B = 1$ iff $A \leq B$
- Residuation: $A \otimes B \leq C$ iff $A \leq B \to C$
- Restriction on truth functions of connectives

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T-norm conjunctions

Reasonable restrictions on the truth function of conjunction:

- Commutativity: $A \otimes B = B \otimes A$
- Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
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= *Continuous t-norms*

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T-norm conjunctions

Reasonable restrictions on the truth function of conjunction:

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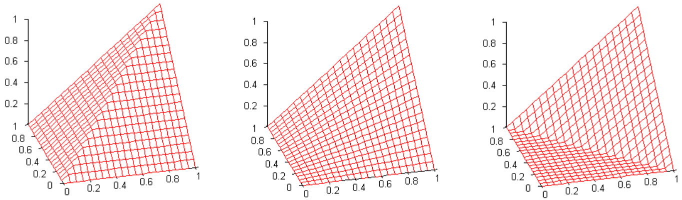
Continuous t-norms

Salient examples on $[0,1]$:

All continuous t-norms are ordinal sums of these three:

$A \otimes_G B = \min(A, B)$	Gödel t-norm
$A \otimes_\Pi B = A \cdot B$	product t-norm
$A \otimes_Ł B = \max(A + B - 1, 0)$	Łukasiewicz t-norm

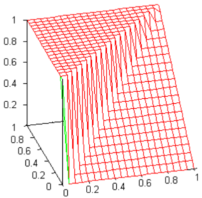


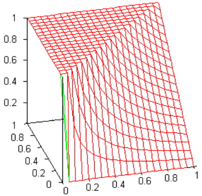
The residua of continuous t-norms

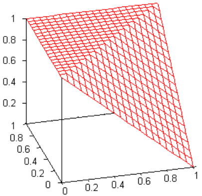
Their residua (uniquely defined by the residuation condition):

$A \rightarrow B = 1$ if $A \leq B$, otherwise:

$$A \rightarrow_G B = B$$
$$A \rightarrow_\Pi B = B/A$$
$$A \rightarrow_Ł B = \min(1 - A + B, 1)$$







Logics of continuous t-norms

Connectives:

- Conjunction = a continuous t-norm
- Implication = its residuum
- Negation = $A \rightarrow 0$ (reductio ad absurdum)
- Disjunction = max
- Min-conjunction = min

Logic:

Tautologies = always evaluated to 1
Entailment = preservation of the truth degree 1

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T-norm fuzzy logics

Gödel, Łukasiewicz, product fuzzy logic (of respective t-norms)

Fuzzy logic BL = truth under *all* continuous t-norms

Variations: discarding conditions (MTL), adding conditions, adding connectives

Prelinearity: $(A \to B) \vee (B \to A) = 1$ (G = Int + prelinearity)

Logics suitable for graduality
(vs vagueness = graduality + indeterminacy)

Axiomatization: tautologies finitely axiomatizable, entailment seldom (finitary companions only)

Intended vs general semantics (semilinearity)

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Lattice quantifiers

$\forall$, $\exists$ = infimum, supremum (in the lattice of truth values)

Rasiowa's axioms for quantifiers: specification, dual specification, quantifier shifts, generalization

First-order t-norm fuzzy logics: $G\forall, Ł\forall, \Pi\forall, BL\forall, \ldots$

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Generalized quantifiers

Strong quantifiers (corresponding to strong conjunction)

Linguistic quantifiers (*most, almost all, ...*)

Can be modeled in higher-order fuzzy logics, partly an open problem

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- Axiomatization, completeness
- Matrix semantics $\Rightarrow$ Algebraic Logic, Abstract Algebraic Logic
- Functional completeness, functional representation
- Decidability, computational complexity
- Metatheorems analogous to classical logic (deduction, compactness, interpolation, etc)

Example: the Deduction–Detachment Theorem

In classical logic: $\Gamma, A \vdash B$ iff $\Gamma \vdash A \to B$

In Ł$_3$: $\Gamma, A \vdash B$ iff $\Gamma \vdash (A \otimes A) \to B$

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Some answers: usefulness, extension, the metalevel *is* crisp

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Single intended matrix semantics vs other kinds of semantics (Int/modal/. . . /$Bool_{2^n}$)

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Literature

- Standford Encyclopedia of Philosophy
- Handbook of Philosophical Logic
- Gottwald: A Treatise on Many-Valued Logic
- Mleziva: Neklasické logiky
- Peregrin: Logika a logiky