IA159 Formal Methods for Software Analysis

Deductive Software Verification

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Focus and sources

focus

- first formal approach to verification of algorithms and computer programs
- partial and total correctness
- formal system for verification of flowcharts by Floyd (1967)
- axiomatic program verification by Hoare (1969)

source

- Chapter 7 of *D. A. Peled: Software Reliability Methods, Springer, 2001.*

Assumptions and basic terminology

- for simplicity we consider only deterministic programs where the initial values of a program are stored in input variables $x0, x1, \ldots$ and these variables do not change their values during any execution of the program
- a state of a program is an assignment to the program variables
- given a program P and its states a, b, by $P(a, b)$ we denote the fact that the execution of P starting from the state a terminates with the state b
- $a \models \varphi$ denotes that the state a satisfies the formula φ

Terminology

in this lecture, a specification (or a desired property) of a program P is given by two first order formulae

- initial condition φ is a formula with free variables among input variables of P
- final assertion ψ

Two notions of correctness

Definition (partial correctness)

A program P is partially correct with respect to φ and ψ, written $\{\varphi\}P\{\psi\}$, iff for all states a, b it holds

$$P(a, b) \wedge a \models \varphi \implies b \models \psi.$$

Intuitiely, if the program starts with a state satisfying φ and then terminates, then the terminal state satisfies ψ.

Two notions of correctness

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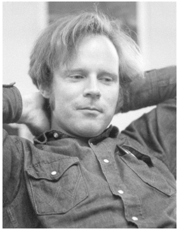
Definition (total correctness)

A program P is totally correct with respect to φ and ψ, written $\langle\varphi\rangle P\langle\psi\rangle$, iff $\{\varphi\}P\{\psi\}$ and for every state a satisfying φ the program terminates.

Intuitiely, if the program starts with a state satisfying φ, then it terminates and the terminal state satisfies ψ.

Formal system for verification of flowcharts



by Robert W Floyd (1936–2001)

- 1965: associate professor at Carnegie–Mellon University
- 1968: full professor at Stanford University, without Ph.D.
- Floyd–Warshall algorithm: shortest paths in a graph
- Floyd–Steinberg dithering: rendering images
- program verification, parsing, sorting

Flowcharts: four kinds of nodes

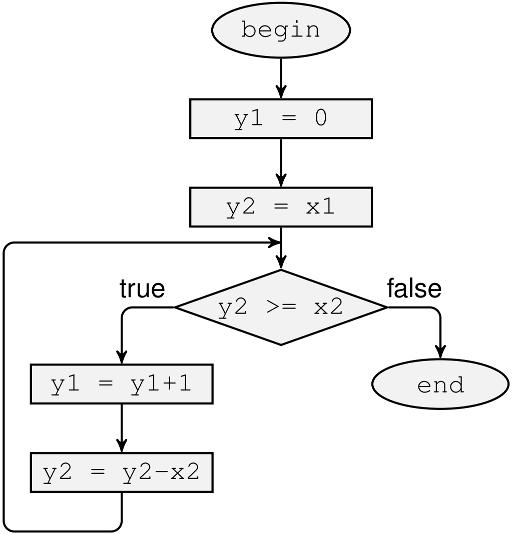
begin one outgoing edge, no incoming edges

end one incoming edge, no outgoing edges

assignment $v = e$, where v is a variable, e is a first order term;
one or more incoming edges, one outgoing edge

decision predicate p is a quantifier-free first order formula;
one or more incoming edges, two outgoing edges marked with true and false

Example: what is this program good for?



initial condition

$\varphi \equiv x1 \geq 0 \;\wedge\; x2 > 0$

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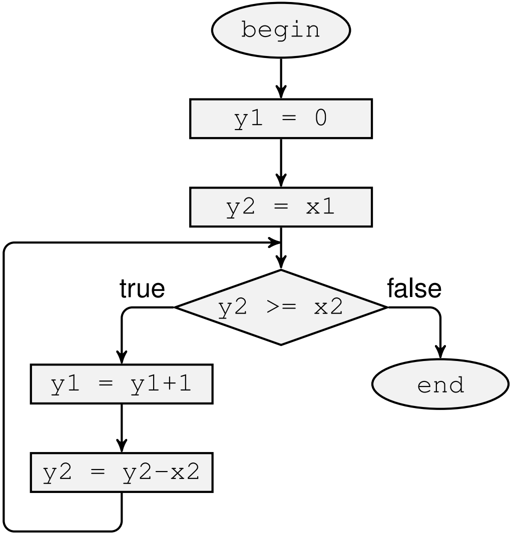

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$\varphi \equiv x1 \geq 0 \;\wedge\; x2 > 0$

final assertion

$$\psi \equiv (x1 = y1 * x2 + y2) \wedge \\ \wedge\; y2 \geq 0 \;\wedge\; y2 < x2$$

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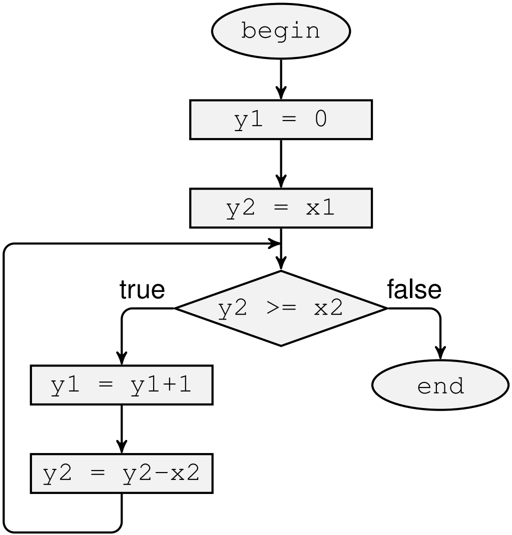

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It computes an integer division.

Formal system for verification of flowcharts

Proving partial correctness

Proving partial correctness

A location of a flowchart program is an edge connecting two flowchart nodes.

To verify that a program P is partially correct with respect to an initial condition φ and a final assertion ψ, it is sufficient to perform the following two steps.

Proving partial correctness: step 1

Step 1

- to each location of the flowchart we attach a first order formula called assertion or invariants
- to the location exiting from begin we attach φ
- to the location entering end we attach ψ

Idea

These assertions should be satisfied by every state reachable in the corresponding location by an execution starting in a state satisfying φ.

Proving partial correctness: step 2

Given an assignment or decision node *c*, every assumption on

- an incoming edge is called precondition, written *pre*(*c*)
- an outgoing edge is called postcondition, written *post*(*c*)

Idea of step 2

We have to prove that whenever the control of the program is just before a node *c* with a state satisfying *pre*(*c*) and execution of *c* moves the control to the location annotated with *post*(*c*), then the state after the move satisfies *post*(*c*).

Proving partial correctness: step 2

Step 2

Every triple $pre(c), c, post(c)$ is treated according to its form.

1. c is a decision node with a predicate p and $post(c)$ is associated to the outgoing edge marked with true.

   Then we need to prove:

   $$pre(c) \land p \implies post(c)$$

2. c is a decision node with a predicate p and $post(c)$ is associated to the outgoing edge marked with false.

   Then we need to prove:

   $$pre(c) \land \neg p \implies post(c)$$

Proving partial correctness: step 2

3. c is an assignment of the form $v = e$, where v is a variable and e an expression.

   The states before and after the assignment are different (i.e. $pre(c)$ and $post(c)$ reason about different states). Therefore, we relativize the postcondition to assert about the states before the assignment.

   Hence, we have to prove

$$pre(c) \implies post(c)[v/e]$$

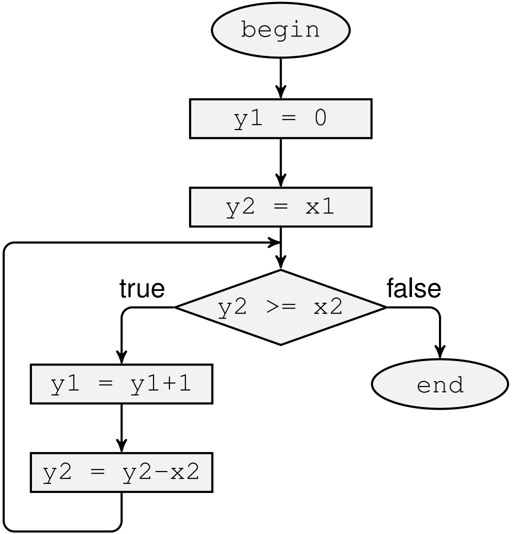
   where $post(c)[v/e]$ is the assertion $post(c)$ where all occurrences of v are replaced with e.

Proving partial correctness

- proving the consistency between each precondition and postcondition of all nodes guarantees that $\{\varphi\}P\{\psi\}$
- in fact, it guarantees even a stronger property:

  In each execution that starts with a state satisfying the initial condition of the program, when the control of the program is at some location, the assumption attached to that location holds.

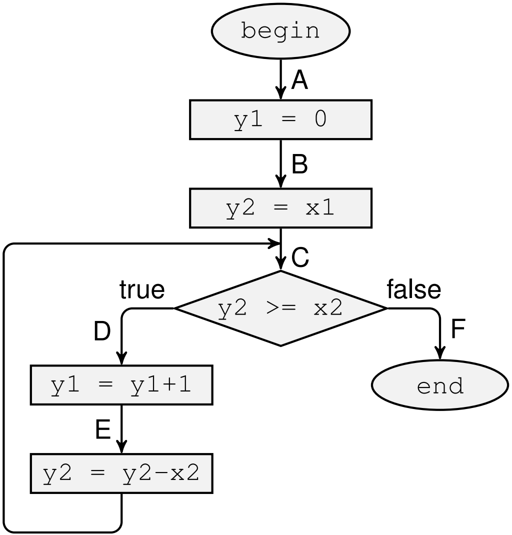
Example: partial correctness



$$\varphi \equiv x1 \geq 0 \ \wedge \ x2 > 0$$

$$\psi \equiv (x1 = y1 * x2 + y2) \wedge$$
$$\wedge\, y2 \geq 0 \ \wedge \ y2 < x2$$

Example: partial correctness



$$\varphi \equiv x1 \geq 0 \wedge x2 > 0$$

$$\psi \equiv (x1 = y1 * x2 + y2) \wedge \wedge y2 \geq 0 \wedge y2 < x2$$

$$\varphi_A \equiv \varphi$$

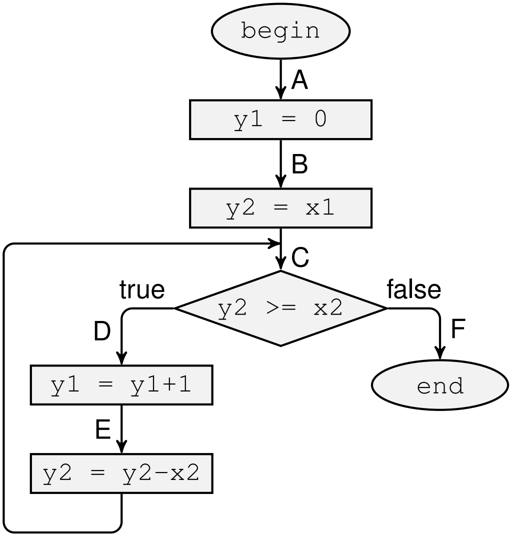
$$\varphi_B \equiv x1 \geq 0 \wedge x2 > 0 \wedge y1 = 0$$

$$\varphi_C \equiv (x1 = y1 * x2 + y2) \wedge \wedge y2 \geq 0$$

$$\varphi_D \equiv (x1 = y1 * x2 + y2) \wedge \wedge y2 \geq x2$$

$$\varphi_E \equiv (x1 = y1 * x2 + y2 - x2) \wedge \wedge y2 - x2 \geq 0$$

$$\varphi_F \equiv \psi$$



$$\varphi \equiv x1 \geq 0 \ \wedge \ x2 > 0$$

$$\psi \equiv (x1 = y1 * x2 + y2) \wedge \wedge\, y2 \geq 0 \ \wedge \ y2 < x2$$

$$\varphi_A \equiv \varphi$$

$$\varphi_B \equiv x1 \geq 0 \ \wedge \ x2 > 0 \ \wedge \ y1 = 0$$

$$\varphi_C \equiv (x1 = y1 * x2 + y2) \wedge \wedge\, y2 \geq 0$$

$$\varphi_D \equiv (x1 = y1 * x2 + y2) \wedge \wedge\, y2 \geq x2$$

$$\varphi_E \equiv (x1 = y1 * x2 + y2 - x2) \wedge \wedge\, y2 - x2 \geq 0$$

$$\varphi_F \equiv \psi$$

Step 2: check the consistency

Notes

- finding assertions for the proof may be a difficult task
- there are some heuristics and tools suggesting invariants
- there cannot be a fully automatic way of finding them (the problem is undecidable)
- in some programming languages, assertions can be inserted into the code as additional runtime checks so that the program will break with a warning message whenever an invariant is violated

Programs with array variables: a problem

Example

- precondition $pre(c) \equiv x[1] = 1 \wedge x[2] = 3$
- assignment $x[x[1]] = 2$
- postcondition $post(c) \equiv x[x[1]] = 2$
- it is easy to prove
  $$pre(c) \implies post(c)[x[x[1]]/2]$$
  as $post(c)[x[x[1]]/2]$ is in fact $2 = 2$
- but if $pre(c)$ holds and the assignment is performed, then $x[1] = 2$ and $x[x[1]] = 3$ and $post(c)$ does not hold

To handle programs with array variables, the method has to be modified in one point: relativization of postconditions of assignment nodes.

Modification for array variables

- let x be an array variable and $e1, e2, e3$ terms
- the syntax of terms is extended with a new construct $(x; e1{:}e2)[e3]$, where $(x; e1{:}e2)$ represents almost the same array as x, only the element with the index $e1$ has been set to $e2$
- to check the consistency of an assignment $x[e1] = e2$ with a precondition $pre(c)$ and postcondition $post(c)$, we have to prove

$$pre(c) \implies post(c)[x/(x; e1{:}e2)]$$

- the added construct does not increase the expressiveness of the logic: a formula ρ containing $(x; e1{:}e2)[e3]$ can be translated into an equivalent formula

$$(e1 = e3 \land \rho[(x; e1{:}e2)[e3]/e2]) \lor$$
$$\lor (\neg(e1 = e3) \land \rho[(x; e1{:}e2)[e3]/x[e3]])$$

Proving termination

Proving termination: terminology

- a partially ordered domain is a pair $(W, \prec)$ where W is a set and $\prec$ is a strict partial order relation over W (i.e. irreflexive, asymmetric, and transitive)
- $u \succ v$ has the same meaning as $v \prec u$
- we denote $u \succeq v$ when $u \succ v$ or $u = v$
- a well founded domain is a partially ordered domain containing no infinite sequence of the form

  $$w_0 \succ w_1 \succ w_2 \succ w_3 \succ \dots$$

  (i.e. no infinite decreasing sequence)

Proving termination

To prove the termination with respect to the initial condition φ, we need to do the following steps.

1. We select a well founded domain $(W, \prec)$ such that W is a subset of the domain of program variables and $\prec$ is expressible using the signature of the program.
2. To each location in the flowchart we attach an invariant and an expression. To the location exiting from begin we attach φ.
3. We show the consistency for each triple $pre(c), c, post(c)$, as in the partial correctness proof.

Proving termination

4. We show that whenever an execution starting in a state satisfying φ reaches some location, the value of the expression associated to this location is within W.

   Formally, we prove that for each location with the associated invariant ρ and expression e it holds:

$$\rho \implies (e \in W)$$

   Note that $e \in W$ is not, in general, a first order logic formula. However, it can often be translated into a first order formula.

Proving termination

5 We show that in each execution of the program, when proceeding from a location to its successor location, the value of the associated expressions does not increase.

Formally, for every node c, an incoming edge with the associated invariant $pre(c)$ and expression $e1$, and an outgoing edge with the associated expression $e2$

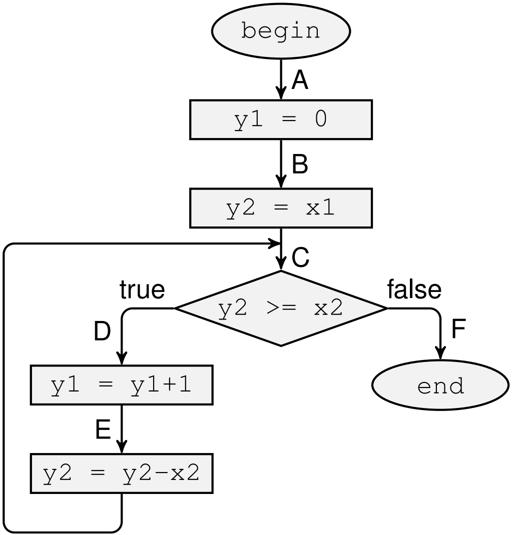
- if c is a decision node with a predicate p and $e2$ is associated with the true edge, then we prove:
$$pre(c) \land p \implies e1 \succeq e2$$
- if c is a decision node with a predicate p and $e2$ is associated with the false edge, then we prove:
$$pre(c) \land \neg p \implies e1 \succeq e2$$
- if c is an assignment $v = e$, then we prove:
$$pre(c) \implies e1 \succeq e2[v/e]$$

Proving termination

6. In each execution of the program, during each traversal of a cycle (a loop) in the flowchart there is some point where a decrease occurs in the value of the associated expressions from one location to its successor.

   Formally, for each path through any cycle we have to find a node with an incoming and an outgoing edge such that the corresponding implication above holds even if $\succeq$ is replaced with $\succ$.

Example: termination



initial condition

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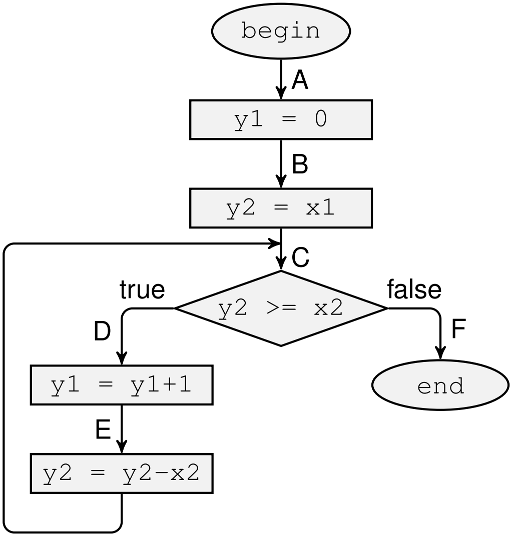
Example: termination



initial condition

$\varphi \equiv x1 \geq 0 \;\wedge\; x2 > 0$

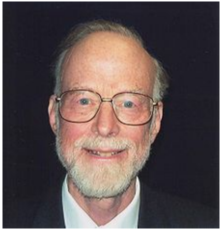
$$
\begin{aligned}
\varphi_A &\equiv \varphi \\
\varphi_B &\equiv x1 \geq 0 \wedge x2 > 0 \\
\varphi_C &\equiv x2 > 0 \wedge y2 \geq 0 \\
\varphi_D &\equiv x2 > 0 \wedge y2 \geq x2 \\
\varphi_E &\equiv x2 > 0 \wedge y2 \geq x2 \\
\varphi_F &\equiv y2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
e_A &= x1 \\
e_B &= x1 \\
e_C &= y2 \\
e_D &= y2 \\
e_E &= y2 \\
e_F &= y2
\end{aligned}
$$

Notes

- it may be difficult to find the right well founded domain, invariants, and expressions
- termination and partial correctness can be proven simultaneously

Axiomatic program verification



by sir Charles Antony Richard Hoare (1934)

- studied in Oxford University and Moscow State University
- Quicksort algorithm (1960)
- Hoare logic: program verification
- Communicating Sequential Processes (CSP)
- null pointer
- now in Microsoft Research

Hoare logic

- a proof system that includes both logic and pieces of code
- allows to prove different sequential parts of the program separately (and combine the proofs later)
- constructed on top of some first order deduction system

Hoare logic

- contains Hoare triples of the form $\{\varphi\}S\{\psi\}$, where φ, ψ are first order formulae and S is (a part of) a program with the syntax:

$$S ::= \; v = e \mid \textit{skip} \mid S; S \mid \textit{if } p \textit{ then } S \textit{ else } S \textit{ fi} \mid$$
$$\textit{while } p \textit{ do } S \textit{ end} \mid \textit{begin } S \textit{ end}$$

  where v is a variable, e is a first order expression, and p is a quantifier-free first order formula
- a Hoare triple $\{\varphi\}S\{\psi\}$ means that if an execution of S starts with a state satisfying φ and S terminates from that state, then a state satisfying ψ is reached
- if S is the entire program, then $\{\varphi\}S\{\psi\}$ claims that S is partially correct with respect to initial condition φ and final assertion ψ

Axioms and proof rules

Assignment axiom

$$\{\varphi[v/e]\}v = e\{\varphi\}$$

Skip axiom

$$\{\varphi\}skip\{\varphi\}$$

Left strengthening rule

$$\frac{\varphi \implies \varphi' \quad \{\varphi'\}S\{\psi\}}{\{\varphi\}S\{\psi\}}$$

Right weakening rule

$$\frac{\{\varphi\}S\{\psi'\} \quad \psi' \implies \psi}{\{\varphi\}S\{\psi\}}$$

Axioms and proof rules

Sequential composition rule

$$\frac{\{\varphi\}S_1\{\eta\} \quad \{\eta\}S_2\{\psi\}}{\{\varphi\}S_1; S_2\{\psi\}}$$

If-then-else rule

$$\frac{\{\varphi \land p\}S_1\{\psi\} \quad \{\varphi \land \neg p\}S_2\{\psi\}}{\{\varphi\}\mathit{if}\ p\ \mathit{then}\ S_1\ \mathit{else}\ S_2\ \mathit{fi}\{\psi\}}$$

While rule

$$\frac{\{\varphi \land p\}S\{\varphi\}}{\{\varphi\}\mathit{while}\ p\ \mathit{do}\ S\ \mathit{end}\{\varphi \land \neg p\}}$$

Begin-end rule

$$\frac{\{\varphi\}S\{\psi\}}{\{\varphi\}\mathit{begin}\ S\ \mathit{end}\{\psi\}}$$

Derived rules

Assignment axiom + left strengthening rule

$$\frac{\varphi \implies \psi[v/e] \quad \{\psi[v/e]\}v = e\{\psi\}\ \textit{(axiom)}}{\{\varphi\}v = e\{\psi\}}$$

Sequential composition + right weakening rule

$$\frac{\{\psi\}S_1\{\eta_1\} \quad \eta_1 \implies \eta_2 \quad \{\eta_2\}S_2\{\psi\}}{\{\varphi\}S_1; S_2\{\psi\}}$$

The proof trees are constructed as usual...

Extensions of Hoare logic

Extensions of the Hoare proof system for verifying concurrent programs provide axioms for

- dealing with shared variables
- synchronous and asynchronous communication
- procedure calls

They are usually tailored for a particular programming language, e.g. Pascal or CSP.

Soundness and completeness

- Hoare's proof system is sound.
- It is not complete due to incompleteness of first order logic with natural numbers and basic arithmetic operations over them (Gödel's incompleteness theorem).
- It is relatively complete, i.e. any correct assertion can be proved under the following two (sometimes unrealistic) conditions:
  - Every correct (first order) logic assertion that is needed in the proof is already included as an axiom in the proof system. (Alternatively: there is an oracle (e.g. a human) deciding whether such an assertion is correct or not.)
  - Every invariant and intermediate assertion that we need for the proof can be expressed using the underlying (first order) logic.
- The relative completeness implies that the system is complete for first order logic with natural numbers with addition and subtraction as the only operators.

Notes

deductive verification

- is not limited to finite state systems
- can handle programs of various domains and datastructures (and even parametrized programs)
- can be applied directly to the code (in principle)
- can verify that the program is correct (but a bug can occur in compiler, in hardware, due to a wrong initial condition or difference between an assumed semantics of code and the real one, etc.)
- is not scalable

Notes

in practice, deductive verification

- needs a great mental effort as it is mostly manual (the result depends strongly on the ingenuity of the people performing verification)
- is significantly slower than the typical speed of effective programming
- is not performed frequently on the actual code (this is changing with new tools)
- can be performed on basic algorithms or on abstractions of the code (the faithfulness of the translation of a program into an abstracted one can sometimes also be formally verified)

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Dafny

- a verification-aware programming language with native support for recording specifications and equipped with a static program verifier
- VSCode plugin available
- look at `https://dafny.org`