Exercise Sheet 1: Games in Strategic Form

Due Date: November 6, 2024

Instructions

- Submit your exercises as a PDF file through the file vault (odevzdávárna) in the IS.
- Solutions will be graded based on their correctness and the *clarity* of the arguments presented.
- You do not have to provide proofs and justifications if you are not explicitly instructed to do so.
- Collaboration: While you may discuss the exercises with classmates, you must write down the solutions on your own.
- The bonus exercise is optional and will not be graded (although feedback will be provided if you submit a solution). Solving the exercise serves mainly your benefit.
- To pass the exercise sheet, you need to obtain **at least 55** points out of 100 possible.
- If you do not meet the threshold for the minimum points, you may resubmit your solution after the first trial is marked.

Exercises

Exercise 1: (max. 25 points)

a) [**5 points**] Formalize the following scenario as a strategic form game involving two players: There are two shepherds, Alfred and Bob. Alfred owns three cows, while Bob owns two. They have the option to either graze their cows on a shared meadow or keep them on their respective farms. If there are N cows grazing in the same location (whether the meadow, Alfred's farm, or Bob's farm), each cow from that location generates the utility $2 - \frac{N}{3}$ (in the form of milk) for its owner. Both shepherds must decide how many of their cows (ranging from 0 to all) they want to send to graze in the shared meadow.

b) [**5 points**] We define an *IDC* (I Don't Care) strategy as one where a shepherd randomly sends a uniformly distributed number of cows, ranging from 0 to the maximum possible, to graze on the shared meadow. What is the expected utility for each player if both adopt the IDC strategy?

c) [**5 points**] Identify all Alfred's best responses if Bob adopts the IDC strategy?

d) [**10 points**] Identify all (mixed and pure) Nash equilibria of the game and justify there are no others. (*Hint: Use concepts from the lecture to find a simple and concise solution. Avoid long explanations.*)

Exercise 2: (max. 25 points)

a) [**10 points**] In this exercise, consider only pure strategies. Find a strategic form game with two players, P_0 and P_1, where all the following conditions are satisfied:
   - P_0 has two strategies ($S_0 = \{A, B\}$), and P_1 has three strategies ($S_1 = \{X, Y, Z\}$).
   - There is exactly one Nash equilibrium.
   - Every strategy profile is Pareto optimal.[1]

[1] A profile s is Pareto optimal if there is no other profile s' such that $u_i(s') \geq u_i(s)$ for all players i, and $u_i(s') > u_i(s)$ for at least one player i.

- Neither player has a weakly dominant strategy.[2]

b) [**8 points**] Prove that any game meeting the conditions in part (a) must contain a weakly dominated strategy[3].

c) [**7 points**] Characterize all games satisfying the conditions in part (a).

Exercise 3: (max. 25 points)
Consider a strategic form game of two players P_0 and P_1 such that P_1 has two strategies. Prove that a pure strategy of P_0 is strictly dominated by a mixed strategy if and only if it is never a best response.

Exercise 4: (max. 25 points)

a) [**10 points**] Determine whether IESDS with **pure strategies** can introduce new pure Nash equilibria in a finite game.

b) [**15 points**] Determine whether IESDS with **pure strategies** can introduce new pure Nash equilibria in an infinite game.

[2] A strategy s_i of player i is weakly dominant if for every strategy s_i' of player i, $u_i(s_i, s_{1-i}) \geq u_i(s_i', s_{1-i})$ for all $s_{1-i} \in S_{1-i}$, and $u_i(s_i, s_{1-i}) > u_i(s_i', s_{1-i})$ for at least one s_{1-i}.

[3] A strategy s_i of player i is weakly dominated if there exists a strategy s_i' of player i such that $u_i(s_i', s_{1-i}) \geq u_i(s_i, s_{1-i})$ for all $s_{1-i} \in S_{1-i}$, and $u_i(s_i', s_{1-i}) > u_i(s_i, s_{1-i})$ for at least one s_{1-i}.

Bonus Exercise: There is a well-known topological result called Brouwer's Fixed-Point Theorem, which asserts that every continuous function from a convex, closed, and bounded set (such as a ball, simplex, or polyhedron) to itself has at least one fixed point. One way to prove this is through a combinatorial argument known as Sperner's Lemma.

Theorem 1 (Brouwer's Fixed-Point Theorem)**.** *Let X be a non-empty, convex, closed, and bounded set in $\mathbb{R}^n$. Then, every continuous function $f : X \to X$ has a fixed point, i.e., there exists $x \in X$ such that $f(x) = x$.*

In our context, we use this theorem to prove the existence of mixed Nash equilibria in finite games. For simplicity, we will focus on two-player strategic form games, though a similar argument applies to any number of players. Prove the following theorem:

Theorem 2. *Every finite two-player strategic form game has a mixed Nash equilibrium.*

The proof should proceed as follows:

1. Observe that the set of strategy profiles can be represented as a subset of $\mathbb{R}^n$ for a suitable n. (How is n related to the game?)
2. For any fixed strategy profile x, show that the set of best-response profiles[4] is non-empty, compact, convex, and closed.
3. Use (2) to construct a *continuous* function $f : X \to X$ (where X is the set of all strategy profiles) that assigns each x a best-response profile.
4. Apply Brouwer's Fixed-Point Theorem to conclude the existence of a mixed Nash equilibrium.

[4]We call a profile x' a best-response profile for x if every strategy with non-zero probability in x is a best response to x.