Algebra I – autumn 2023

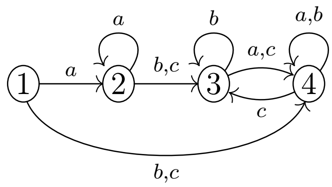
All your assertions should be carefully justified.

1. **(10 points)** Consider the subsets I_1, I_2, I_3 of the ring $\mathbb{Z}[\sqrt{10}][x]$, with I_j consisting exactly of those polynomials whose constant coefficient belongs to the set C_j and whose linear coefficient belongs to the set L_j, where

$$\begin{aligned}
C_1 &= \{\, 10a + b\sqrt{10} \mid a, b \in \mathbb{Z},\ [a]_9 = [b]_9 \,\},\\
L_1 &= \{\, 2c + d\sqrt{10} \mid c, d \in \mathbb{Z},\ [c]_3 = [-d]_3 \,\},\\
C_2 &= \{\, 5a + b\sqrt{10} \mid a, b \in \mathbb{Z},\ [a]_3 = [-b]_3 \,\},\\
L_2 &= \{\, 2c + d\sqrt{10} \mid c, d \in \mathbb{Z} \,\},\\
C_3 = L_3 &= \{\, 5a + b\sqrt{10} \mid a, b \in \mathbb{Z},\ [a]_4 = [b]_4 \,\}.
\end{aligned}$$

   For each of the subsets I_1, I_2, I_3, decide whether it is an ideal of the ring $\mathbb{Z}[\sqrt{10}][x]$.

2. **(10 points)** Determine all elements of the transition monoid of the automaton



3. **(15 points)** Find a direct product of well-known groups that is isomorphic to the quotient group $(G, \cdot)/H$, where

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z},\ c \in \mathbb{Z}[\mathrm{i}] \right\},$$

$$H = \left\{ \begin{pmatrix} 1 & 2d & 2d\,\mathrm{i} + e \\ 0 & 1 & 4d \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, d, e \in \mathbb{Z} \right\}.$$

4. **(10 points)** Find the minimal polynomial of $\sqrt{\sqrt{3}+1} \cdot \mathrm{i} - \sqrt{3} + 1$ over $\mathbb{Q}$.

5. **(15 points)** Express the number

$$\frac{1}{\alpha^3 + \alpha^2 - \alpha}$$

   without using other than rational numbers in denominators, provided that the number α satisfies the equality $\alpha^4 + 3\alpha^3 = 3 \cdot (1 - \alpha^2) \cdot (\alpha + 1)$.

6. **(10 points)** Provide an example of a ring $(R, +, \cdot)$ that is not a subring of any field, and a subring of $(R, +, \cdot)$ that is a field.

7. **(10 points)** Provide an example of a group $(G, \cdot)$, an isomorphism $\varphi\colon G \to G$ and a subgroup $H \subseteq G$ such that $\varphi(H) \subseteq H$ and at the same time $\varphi(H) \neq H$.

8. **(5 points)** What does it mean for an integral domain $(R, +, \cdot)$ to have the unique factorization property?

9. **(5 points)** Describe ideals of polynomial rings over fields. Which of them are such that the corresponding quotient ring is a field/integral domain?

10. **(10 points)** Using only the definitions of groups and homomorphisms, prove that every homomorphism of groups preserves identities and inverses.