Algebra I – autumn 2023

All your assertions should be carefully justified.

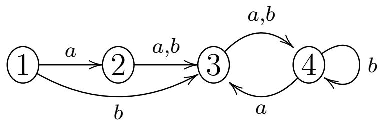
1. **(10 points)** For each of the following three conditions, decide whether it correctly defines a subset I of the ring $(\mathbb{Z}[\sqrt[3]{2}], +, \cdot)$ and whether this subset is an ideal of this ring: Let the set I contain the number $a + b\sqrt[3]{2} + c\sqrt[3]{4} + d\sqrt[3]{16}$, with $a, b, c, d \in \mathbb{Z}$, if and only if

(a) $[a + b + c]_3 = [d]_3$,

(b) $[a + b + d]_3 = [c]_3$,

(c) $[a + c + d]_3 = [b]_3$.

2. **(10 points)** Determine all elements of the transition monoid of the automaton



3. **(15 points)** Find a direct product of well-known groups that is isomorphic to the quotient group $\big((G, \cdot) \times (\mathbb{Z}, +)\big)/H$, where

$$G = \left\{ \begin{pmatrix} p & 0 & 0 \\ f & 1 & 0 \\ h & g & p \end{pmatrix} \;\middle|\; p \in \{1, -1\},\ f, g \in \mathbb{Z}[x],\ h \in \mathbb{Z}[\mathrm{i}][x] \right\},$$

$$H = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ f & 1 & 0 \\ h & g & 1 \end{pmatrix}, f(1) + 2k \right) \in G \times \mathbb{Z} \;\middle|\; k, h(2), h(3) \in \mathbb{Z},\ f(1) + g(1) \text{ is even} \right\}.$$

4. **(10 points)** Find the minimal polynomial of the number $\sqrt[4]{2} + \sqrt[4]{4} + \sqrt[4]{8}$ over $\mathbb{Q}$.

5. **(15 points)** Express the number

$$\frac{1}{\alpha^5 - 2\alpha^4 + \alpha^3 + 3\alpha + 1}$$

without using other than rational numbers in denominators, provided that the number α satisfies the equality $\alpha^3(2 - \alpha) = 2\alpha + 2$.

6. **(10 points)** Provide an example of a finite ring $(R, +, \cdot)$ such that there exist exactly $\frac{9}{10}|R|$ elements $r \in R$ such that some element $s \in R \setminus \{0\}$ satisfies the equality $r \cdot s = 0$.

7. **(10 points)** Provide an example of a group $(G, \cdot)$ and two elements g and h of G such that there exist at least two isomorphisms $\varphi\colon (G, \cdot) \to (G, \cdot)$ and each such isomorphism satisfies both $\varphi(g) = g$ and $\varphi(h) = h$.

8. **(5 points)** Define the field of fractions of an integral domain.

9. **(5 points)** Formulate the theorem that describes irreducible polynomials over the fields $\mathbb{C}$ and $\mathbb{R}$.

10. **(10 points)** Using only the definitions of groups and subgroups, prove that right cosets of a subgroup are pairwise disjoint.