Rotations and quaternions

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PA199

Outline

- Rotation matrix

- Euler angles
- Tait-Bryan angles

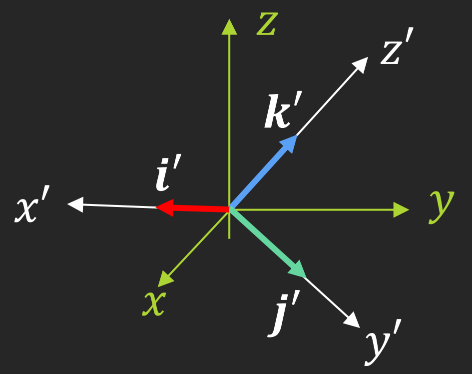
- Axis-angle representation

- Quaternions
  - Rotations via quaternions
  - Quaternion derivative

Rotation matrix

- Well know topic from computer graphics courses.
  => We only discuss relation between basis vectors and rotation matrix.
- Let $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ be **orthonormal** basis vectors of a coordinate system inside the world coordinate system.
- Then, the orientation of the coordinate system is represented by the rotation matrix:

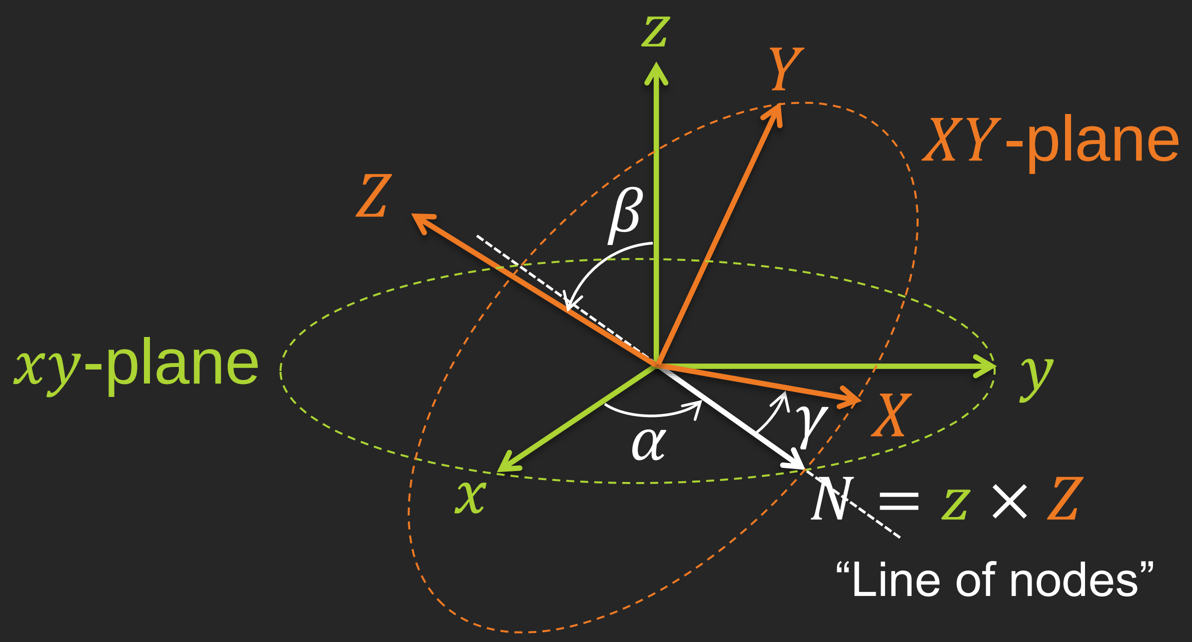
$$R = \begin{pmatrix} \boldsymbol{i}'_x & \boldsymbol{j}'_x & \boldsymbol{k}'_x \\ \boldsymbol{i}'_y & \boldsymbol{j}'_y & \boldsymbol{k}'_y \\ \boldsymbol{i}'_z & \boldsymbol{j}'_z & \boldsymbol{k}'_z \end{pmatrix}$$



- R transforms vectors "to world space".
- $R^{-1} = R^{\mathsf{T}}$ transforms vectors "from world space".

Euler angles

- Three rotations are always sufficient to transform a source frame xyz to a target one XYZ:
  - $\alpha \in \langle 0,2\pi)$
  - $\beta \in \langle 0,\pi\rangle$
  - $\gamma \in \langle 0,2\pi)$
- Actual rotations:



When planes xy and XY are equal, then the statement is clearly true.

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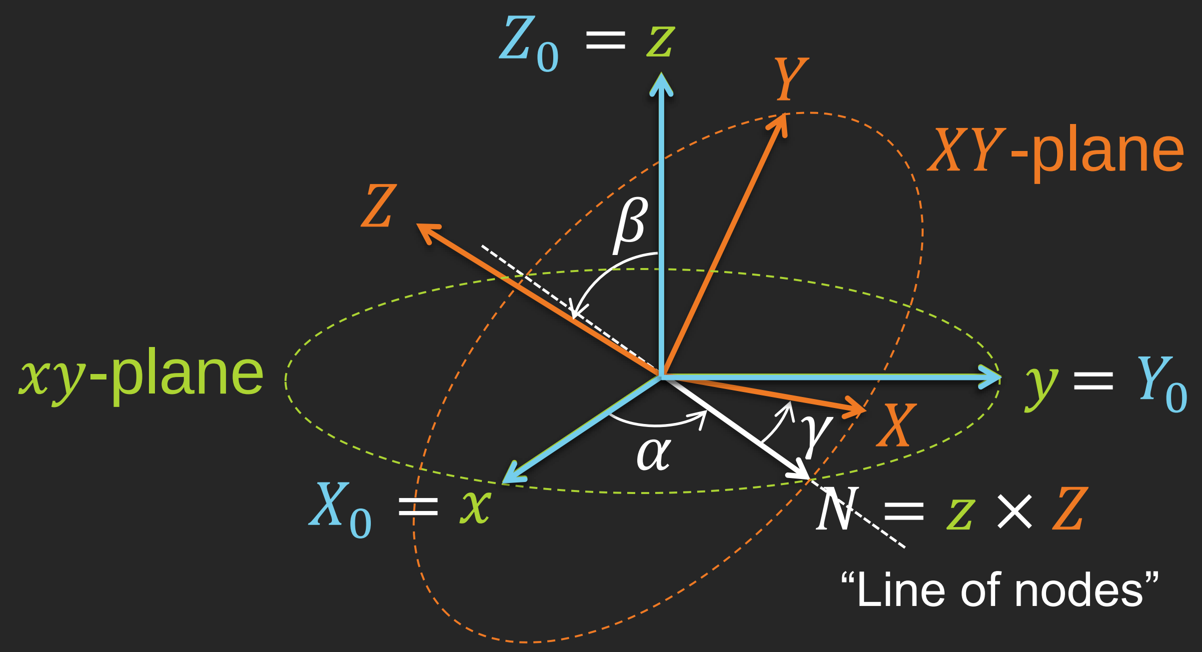
  - Start with frame $X_0Y_0Z_0 = xyz$.
  - Rotate $X_0Y_0Z_0$ about Z_0 by α.



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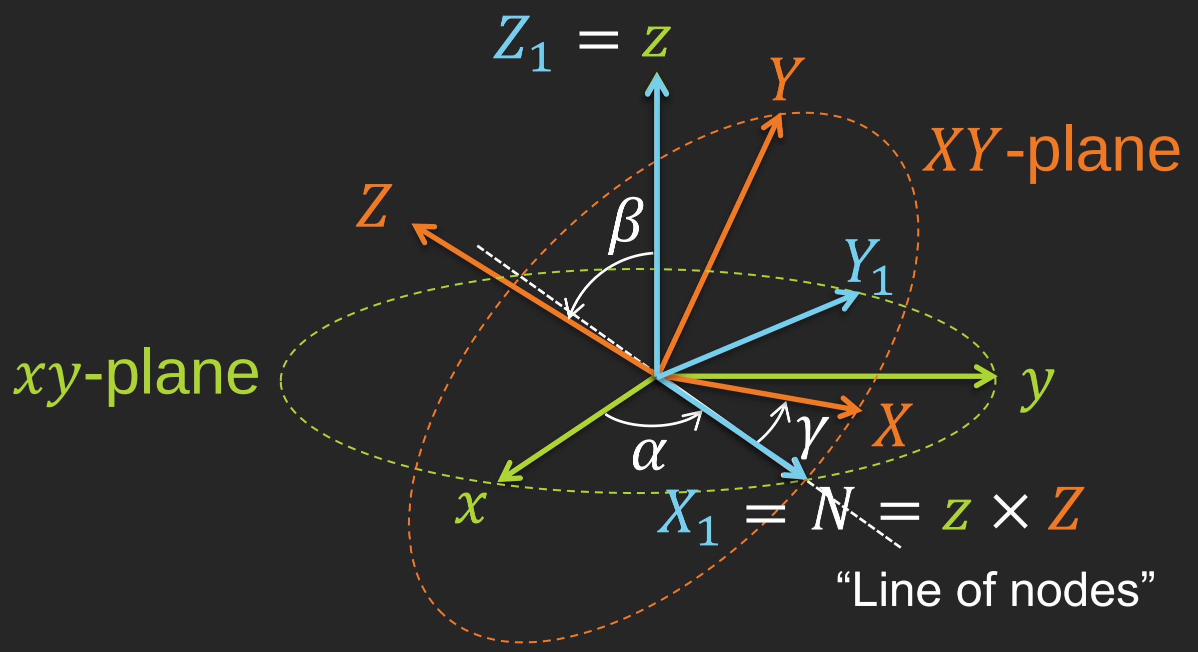
  - Rotate $X_1Y_1Z_1$ about X_1 by β.



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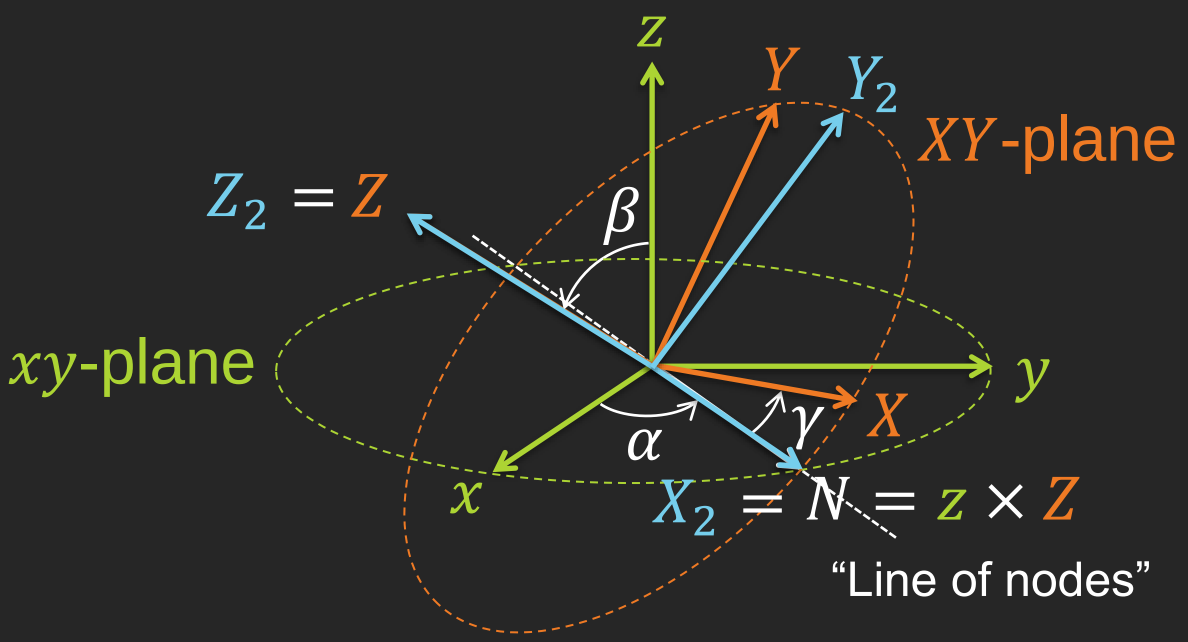
  - Rotate $X_2Y_2Z_2$ about Z_2 by γ.



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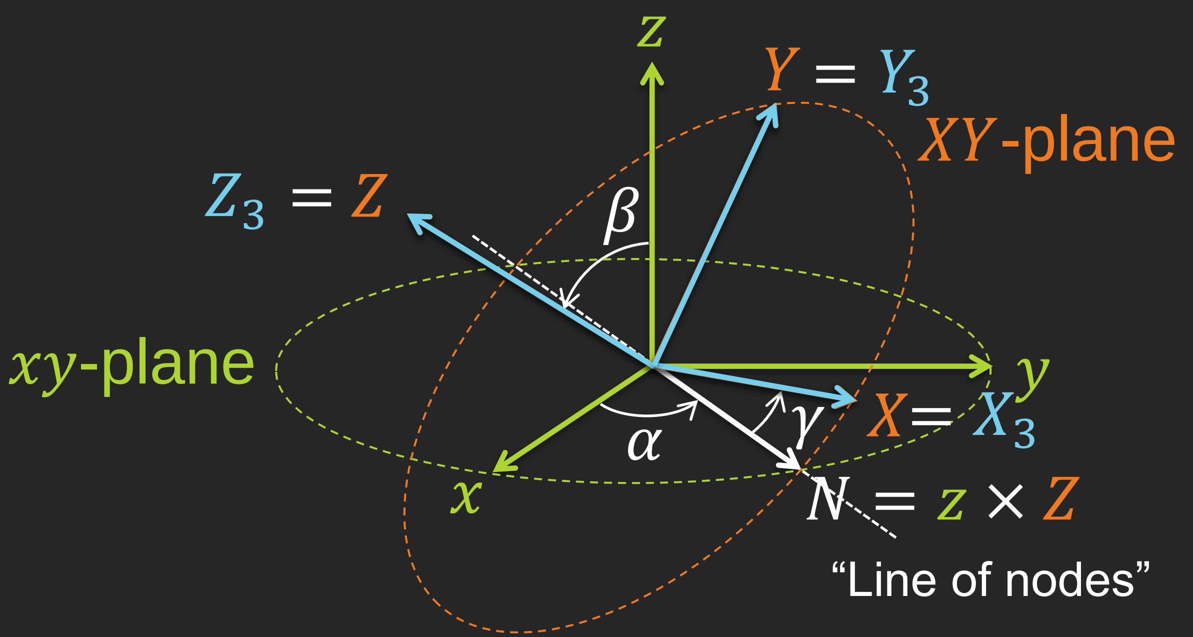


When planes xy and XY are equal, then the statement is clearly true.

Euler angles

- Let $R(\varphi, a)$ denotes a rotation matrix about an axis a by an angle φ.
- So, our rotations can be expressed by matrices:
  - $R(\alpha, Z_0)$
  - $R(\beta, X_1)$
  - $R(\gamma, Z_2)$
- We compose them by the matrix multiplication:

$$R(\gamma, Z_2)R(\beta, X_1)R(\alpha, Z_0)$$

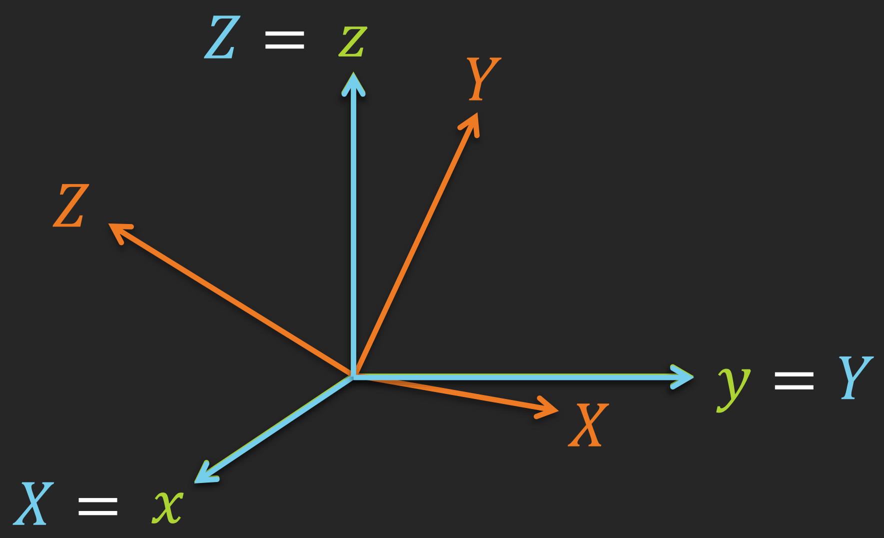
- Here we work with Z-X-Z **convention**. But there are 5 more:
  - X-Y-X, X-Z-X, Y-X-Y, Y-Z-Y, and Z-Y-Z.
  - We can choose any of the conventions we want.
  - Observation: 1^{st} and 3^{rd} rotation axes are the same.

Euler angles

- The rotations $R(\alpha, Z_0)$, $R(\beta, X_1)$, $R(\gamma, Z_2)$ about the axes of the **rotated** (target) frame XYZ are called **intrinsic**.
- A **practical disadvantage** of intrinsic rotations is that some of rotations are about **arbitrary oriented axis**.
  - In CG courses we only learned how to build rotation matrices for fixed axes x, y, z.
  - But axes X_1, Z_2 may be arbitrary (the axis Z_0 is OK, since $Z_0 = z$).
- Fortunately, we can also transform a source frame xyz to a target one XYZ using **extrinsic** rotations $R(\alpha, z)$, $R(\beta, x)$, $R(\gamma, z)$.
  - Let us figure out how to do that…

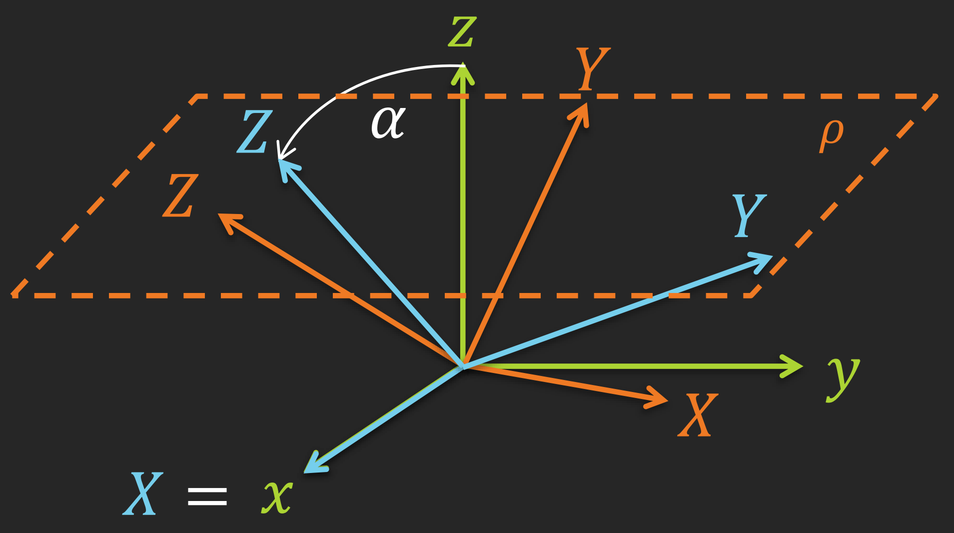
Euler angles

- Start with the XYZ aligned with xyz.



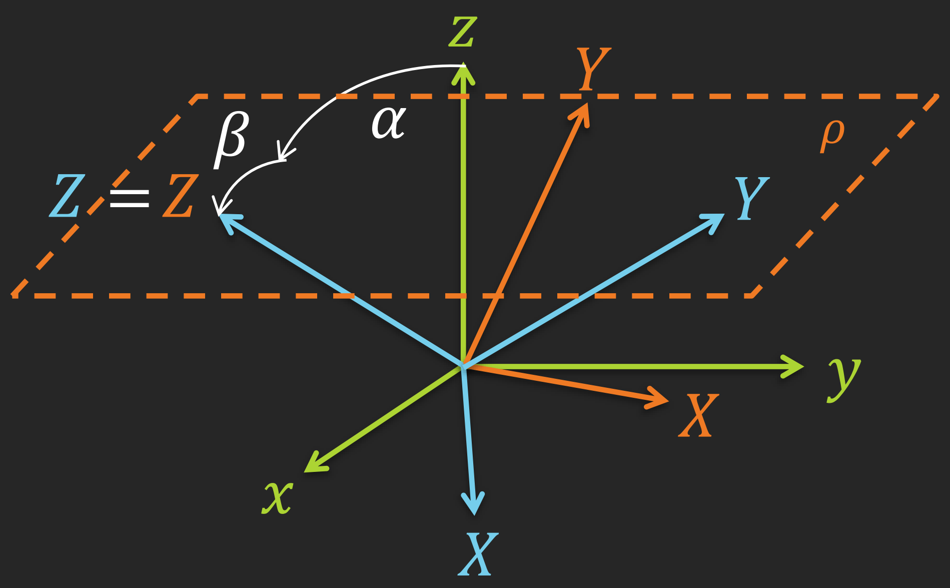
Euler angles

- Start with the XYZ aligned with xyz.
- Apply $R(\alpha, x)$ to rotate the tip of Z to the plane ρ.
  - ρ is parallel with xy plane and contains the tip of Z.



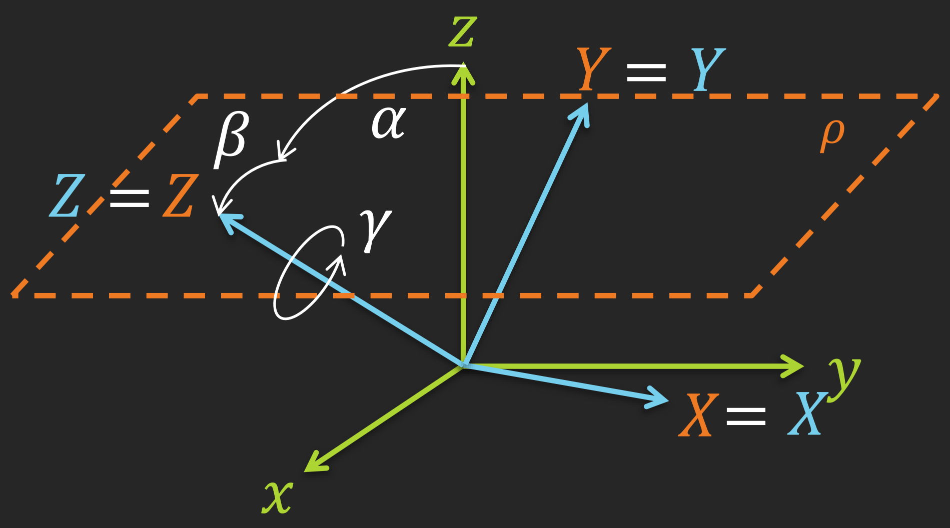
Euler angles

- Start with the XYZ aligned with xyz.
- Apply $R(\alpha, x)$ to rotate the tip of Z to the plane ρ.
  - ρ is parallel with xy plane and contains the tip of Z.
- Apply $R(\beta, z)$ to rotate the tip of Z to the tip of Z.



Euler angles

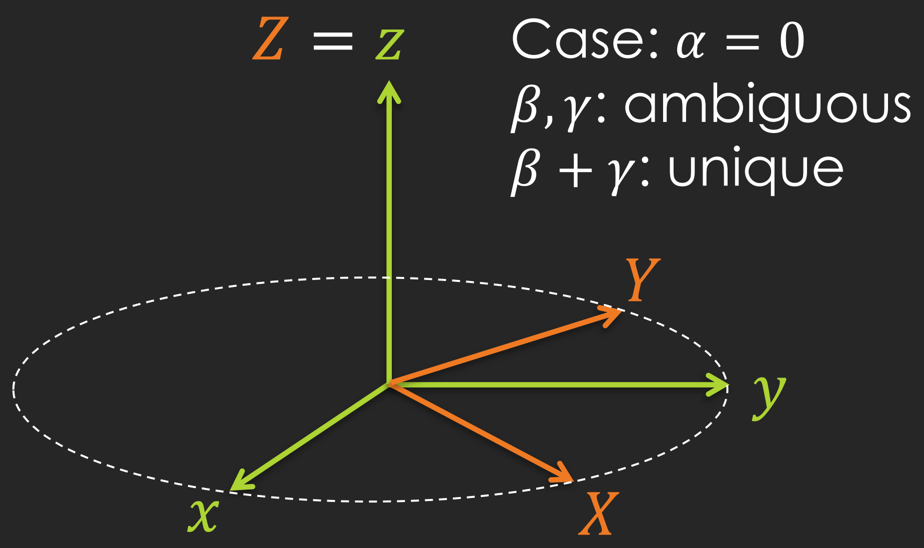
- Start with the XYZ aligned with xyz.
- Apply $R(\alpha, x)$ to rotate the tip of Z to the plane ρ.
  - ρ is parallel with xy plane and contains the tip of Z.
- Apply $R(\beta, z)$ to rotate the tip of Z to the tip of Z.
- Apply the "twist" rotation $R(\gamma, Z)$ to align X with X and Y with Y.
  - But, this is not extrinsic rotation!
  - We can fix it by applying the twist $R(\gamma, z)$ as the first rotation.
  - The value of γ will be different since the other two rotations affect the twist too.

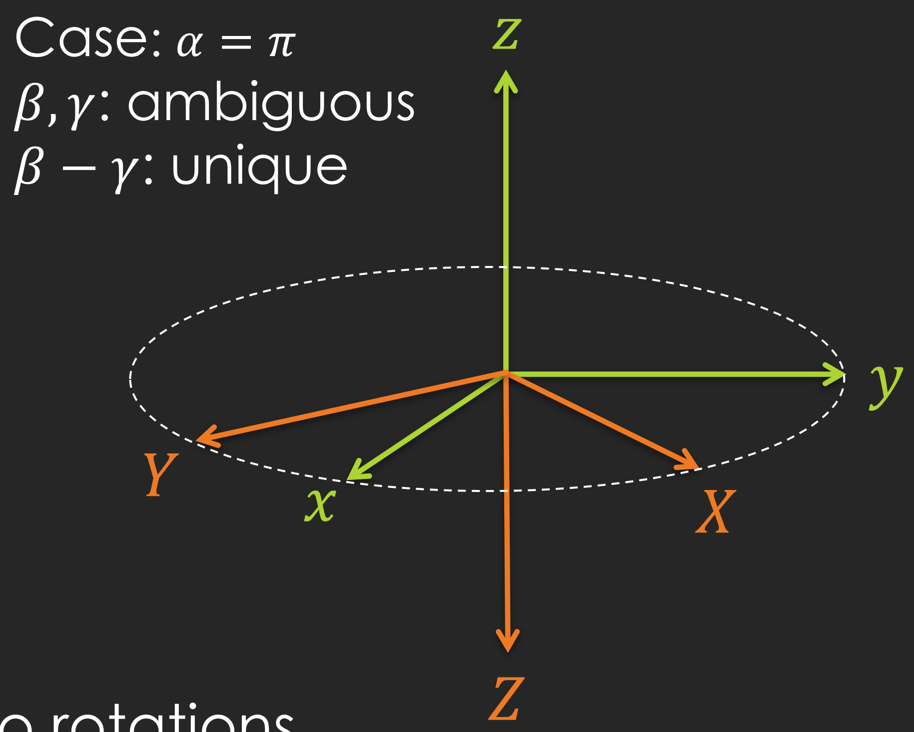


More intuition is in video: [5]

Euler angles: gimbal lock

- Planes xy and XY are parallel => 1 degree of freedom is lost = **gimbal lock**.





- The special value of α “locks” the other two rotations into the same plane (although they can rotate freely in that plane).

Euler angles representation

- We can use 3 angles to express any orientation of an object in 3D space:

```
public class Orientation {
    float alpha;
    float beta;
    float gamma;
};
```

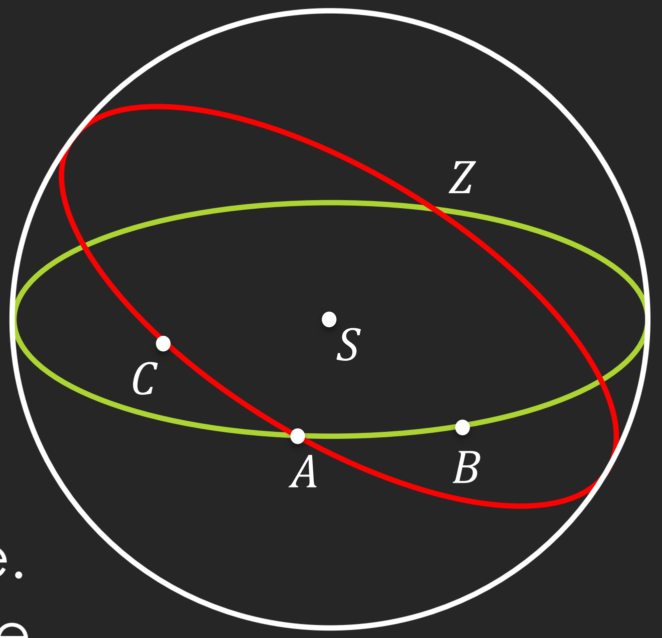
- Pros:
  - Low memory footprint.
  - Easy to understand.
- Cons:
  - Suffers from the gimbal lock.
  - Slow conversion to matrix representation (sin and cos for each angle).

Tait-Bryan angles

- Same as Euler angles, except that **all three axes are different**.
- There are 6 possible **conventions**:
  - X-Y-Z, X-Z-Y, Y-X-Z, Y-Z-X, Z-X-Y, and Z-Y-X.
- The **line of nodes** is different: It is an intersection of the xy-plane and the plane orthogonal to the 3rd rotation axis of the convention.
- The angles α, β, γ are often called yaw, pitch, roll, respectively.

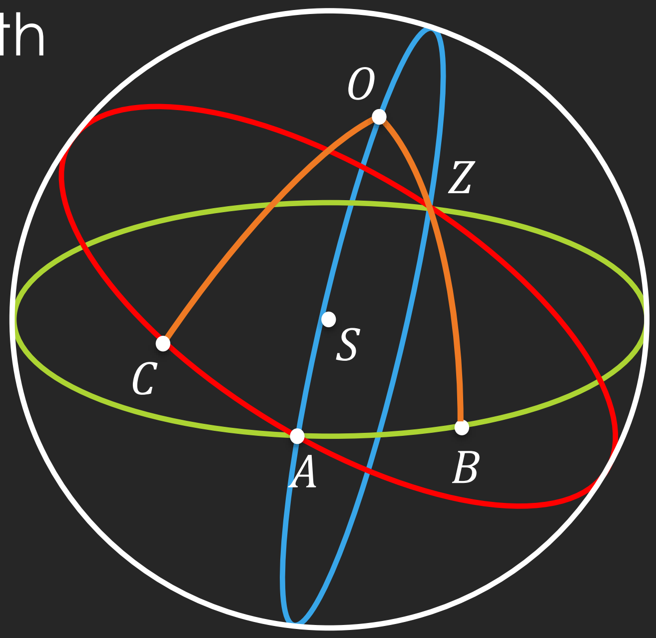
Axis-angle rotation

- **Euler's rotation theorem** (one of the versions): Any reconfiguration of an object in 3D space with one of its points fixed is equivalent to its single rotation about an axis passing through the fixed point.
- Proof:
  - We look for a rotation axis passing though S.
  - We "paint" a great circle (green) on the sphere in the initial position.
  - We rotate the sphere => We get the rotated green circle, which is depicted as red circle.
  - If the circles coincide, then the axis clearly exists. Otherwise, the circles intersect - two points A, Z.
  - A is on red circle => its pre-image B is on green one. A is on green circle => its post-image C is on red one.



Axis-angle rotation

- Construct a great circle (blue) passing through A, Z and bisecting the angle BAC.
- Find a point O on the blue circle s.t. the length of arcs AO and BO is the same.
- The length of the arc AO must be equal to the length of the arc CO, because lengths of arcs AB and AC are the same and the blue circle in the bisector of the angle CAB.
  - $\Rightarrow$ Triangles CAO and ABO on the sphere must be the same.
    Actually, ABO becomes CAO after the rotation.
  - $\Rightarrow$ The point O lies on the searched rotation axis, because it does not move when rotating the triangles.
  - $\Rightarrow$ SO is the rotation axis and the arc length AB is the angle.



Axis-angle rotation

- **Rodrigues' rotation formula**: A vector $v \in R^3$ rotated about a **unit** axis $a \in R^3$ by an angle $\theta \in \langle 0, 2\pi)$ is the vector:

$$\bar{v} = \cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v.$$

  - Proof:

$$v_a = (a \cdot v)a,$$

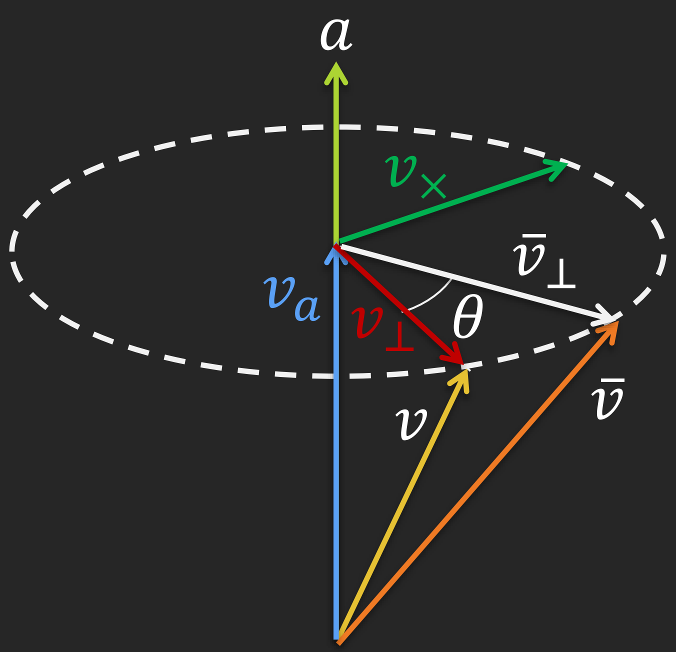
$$v_\perp = v - v_a = v - (a \cdot v)a,$$

$$v_\times = a \times v_\perp = a \times (v - v_a) = a \times v.$$

Note: $|v_\times| = |v_\perp|$.

$$\bar{v}_\perp = \cos\theta\, v_\perp + \sin\theta\, v_\times.$$

$$\begin{aligned}\bar{v} &= v_a + \bar{v}_\perp \\ &= v_a + \cos\theta\, v_\perp + \sin\theta\, v_\times \\ &= (a \cdot v)a + \cos\theta\,(v - (a \cdot v)a) + \sin\theta\, a \times v \\ &= \cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v.\end{aligned}$$



Axis-Angle to rotation Matrix

- Vector triple product: $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$
  - If $|a| = 1$, then $a \times (a \times v) = (a \cdot v)a - (a \cdot a)v = (a \cdot v)a - v$
- Matrix representation of the cross product:

$$u \times v = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} v = [\![u]\!]v.$$

- Matrix representation of the axis-angle:

$$\begin{aligned}
\bar{v} &= \cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v \\
&= \cos\theta\, v + (1 - \cos\theta)(a \times (a \times v) + v) + \sin\theta\, a \times v \\
&= v + (1 - \cos\theta)a \times (a \times v) + \sin\theta\, a \times v \\
&= v + (1 - \cos\theta)[\![a]\!]^2 v + \sin\theta\, [\![a]\!]v \\
&= (I + (1 - \cos\theta)[\![a]\!]^2 + \sin\theta\, [\![a]\!])v \\
&= R(\theta, a)v
\end{aligned}$$

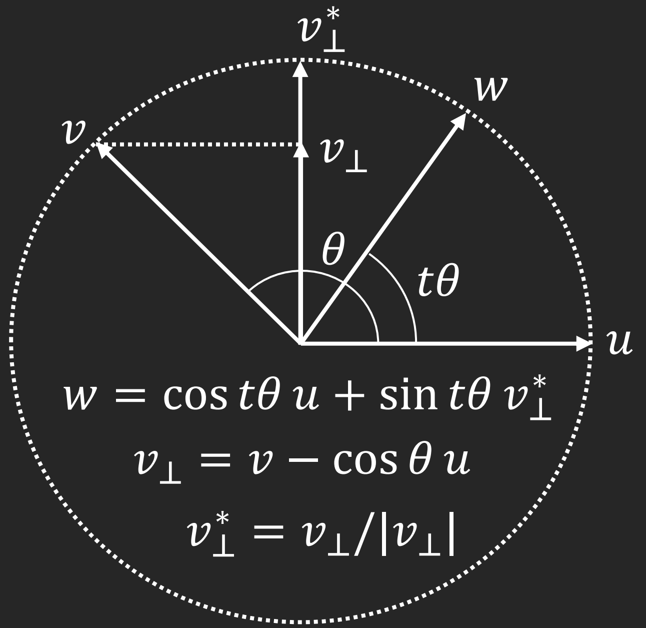
Linear interpolation (lerp)

- Given two axis-angle rotations $R(\varphi, a)$ and $R(\psi, b)$, the linearly interpolated rotation is then $R\left((1-t)\varphi + t\psi, \frac{(1-t)a+tb}{|(1-t)a+tb|}\right)$, $t \in \langle 0,1 \rangle$.
- Technical issues related to $|(1-t)a + tb|$:
  - Slow – we must compute the square root.
  - The result is not defined when =0.

- **Problem**: The velocity is not constant (increases and decreases) .
  $\Rightarrow$ Visible visual artefact – we prefer **uniform** blending between rotations.
- Can we do better?
  - Yes, use **spherical** linear interpolation.

Spherical linear interpolation (slerp)

- Given two **linearly independent unit** vectors u, v and a parameter $t \in \langle 0,1 \rangle$, find a **unit** vector $w = \alpha u + \beta v$ s.t. $\alpha, \beta > 0$ and angle between u, w is $t\theta$, where θ is the angle between u, v.
- $v_\perp^* = \frac{v - \cos\theta u}{\sqrt{(v - \cos\theta u)(v - \cos\theta u)}} = \frac{v - \cos\theta u}{\sqrt{1 - \cos^2\theta}} = \frac{v - \cos\theta u}{\sin\theta}.$
- $$\begin{aligned} w &= \cos t\theta\, u + \sin t\theta \frac{v - \cos\theta u}{\sin\theta} \\ &= \left(\cos t\theta - \frac{\sin t\theta \cos\theta}{\sin\theta}\right) u + \frac{\sin t\theta}{\sin\theta} v \\ &= \frac{\cos t\theta \sin\theta - \sin t\theta \cos\theta}{\sin\theta} u + \frac{\sin t\theta}{\sin\theta} v \\ &= \frac{\sin(1-t)\theta}{\sin\theta} u + \frac{\sin t\theta}{\sin\theta} v. \end{aligned}$$



- Given two axis-angle rotations $R(\varphi, a)$ and $R(\psi, b)$, the interpolated rotation is then $R\left((1-t)\varphi + t\psi, \frac{\sin(1-t)\theta}{\sin\theta} a + \frac{\sin t\theta}{\sin\theta} b\right)$.

Axis-angle representation

- We can use axis-angle to express any orientation of object in 3D space.

```
public class Orientation {
    float angle;
    Vector3 unitAxis;
};
```

- Pros:
  - Fast conversion to matrix representation (sine and cosine for one angle).
  - We can use lerp and slerp.
  - Easy to understand.
  - Low memory footprint.
- Cons:
  - Complicated composition of rotations (often solved via other rep.).

Quaternions

- Let a, b are **real** numbers and $\boldsymbol{i}^2 = -1$ be an **imaginary unit**. Then
$$a + b\boldsymbol{i}$$
is a **complex** number (constructed by the pairing process).
- Let $a + b\boldsymbol{i}, c + d\boldsymbol{i}$ are **complex** numbers and $\boldsymbol{j}^2 = -1$ be an **imaginary unit**, $\boldsymbol{i} \neq \boldsymbol{j}$. Then
$$(a + b\boldsymbol{i}) + (c + d\boldsymbol{i})\boldsymbol{j} =$$
$$a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{i}\boldsymbol{j} =$$
$$a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$$
where $\boldsymbol{k} = \boldsymbol{i}\boldsymbol{j}$, is a **quaternion**.
- $\boldsymbol{k}^2 = -1$ is another unique **imaginary unit**, i.e., $\boldsymbol{k} \neq \boldsymbol{i}, \boldsymbol{k} \neq \boldsymbol{j}$.
- Relations between imaginary units:

$$\boldsymbol{ij} = \boldsymbol{k}, \quad \boldsymbol{jk} = \boldsymbol{i}, \quad \boldsymbol{ki} = \boldsymbol{j},$$
$$\boldsymbol{ji} = -\boldsymbol{k}, \quad \boldsymbol{kj} = -\boldsymbol{i}, \quad \boldsymbol{ik} = -\boldsymbol{j}.$$

How to remember these? Think of the cross product of basis vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, e.g., $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$.

Quaternions

- A quaternion $q = s + u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}$ can be written in a **scalar-vector notation** as a pair $q = (s, u)$, where the vector $u = (u_x, u_y, u_z)^{\mathsf{T}}$.
- Let $q = (s, u), p = (t, v)$ be quaternions and c a real number. Then
  - $q + p = (s, u) + (t, v) = s + u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k} + t + v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} = (s + t, u + v)$.
  - $cq = (cs, cu)$. $(-1)q = -q = (-s, -u)$.
  - $qp = \cdots$ (use distributive law) $\cdots = (st - u \cdot v, sv + tu + u \times v)$. $c(qp) = (cq)p$.
  - Conjugation $q^* = s - u_x\boldsymbol{i} - u_y\boldsymbol{j} - u_z\boldsymbol{k} = (s, -u)$. $cq^* = (cq)^*$.
  - Length $|q| = \sqrt{qq^*} = \sqrt{s^2 + u \cdot u}$. If $|q| = |p| = 1$, then $|qp| = 1$.
  - Additive unit quaternion $(0,0)$, multiplicative unit quaternion $(1,0)$.
  - $q^{-1} = \frac{1}{|q|^2} q^*$. If $|q| = 1$, then $q^{-1} = q^*$.
  - $(q + p)^* = q^* + p^*$.
  - $(qp)^* = p^* q^*$.
  - Dot product: $q \cdot p = st + u \cdot v$.
  - Addition and multiplication are **associative**. Only addition is **commutative**.

Rotation via quaternion

- Let $q = (s, u)$ be a quaternion s.t. $|q| = 1$. Then there exists an angle $\alpha \in \langle 0,2\pi)$ s.t. $q = (\cos\alpha, \sin\alpha\, v)$, where $v = 0$ if $|s| = 1$, else $v = u/\sin\alpha$.
  Proof:
  If $|s| = 1 \;\Rightarrow\; \alpha = 0$. Otherwise,
  $|q| = 1 \;\Rightarrow\; |s| < 1 \;\Rightarrow\; \alpha = \cos^{-1} s$ (choose α s.t. $\sin\alpha > 0$),
  $1^2 = |q|^2 = s^2 + u \cdot u = \cos^2\alpha + |u|^2 \;\Rightarrow\; |u|^2 = 1 - \cos^2\alpha = \sin^2\alpha \;\Rightarrow\; |u| = |\sin\alpha|$.

TADY→

- Let $q = (s, u), p = (0, v)$ be quaternions s.t. $|q| = 1$. Then we define
  $$R_q(p) = qpq^{-1} = qpq^* = \cdots = (0, (s^2 - u \cdot u)v + 2(u \cdot v)u + 2s(u \times v)).$$
- Let us compare the Rodrigues' rotation formula with $R_q(p)$:
  $\cos\theta\, v + (1 - \cos\theta)(a \cdot v)a + \sin\theta\, a \times v.$ ← rotation formula
  $(s^2 - u \cdot u)v + \quad 2(u \cdot v)u + \quad 2s\, u \times v$ ← vector of $R_q(p)$
- **Question**: Is the vector part of $R_q(p)$ equal to Rodrigues' formula?

Rotation via quaternion

- $q = (s, u)$ can be expressed as $q = (\cos\alpha, \sin\alpha\, a), s = \cos\alpha, u = \sin\alpha\, a$.
- Therefore, the vector part of $R_q(p)$is:

$$(s^2 - u \cdot u)v + 2(u \cdot v)u + 2s\, u \times v =$$
$$(\cos^2\alpha - \sin^2\alpha)v + 2\sin^2\alpha\,(a \cdot v)a + 2\cos\alpha\sin\alpha\, a \times v.$$

  So, the angle α must satisfy these three equalities (relevant trigonometry identities are on the right):

$$\cos\theta = \cos^2\alpha - \sin^2\alpha \qquad // \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$1 - \cos\theta = 2\sin^2\alpha \qquad // \cos 2\alpha = 1 - 2\sin^2\alpha$$
$$\sin\theta = 2\cos\alpha\sin\alpha \qquad // \sin 2\alpha = 2\cos\alpha\sin\alpha$$

  A solution exists: $\theta = 2\alpha$. So, $R_q(p)$ rotates p about axis a by 2α.
- Observations, for quaternions q, q', p s.t. $|q| = |q'| = 1$:
  - $q^* R_q(p) q = q^*(qpq^*)q = p$. => q^* is the **inverse** rotation to q.
  - $R_{-q}(p) = (-q)p(-q)^* = qpq^* = R_q(p)$. => $-q$ is the **same** rotation as q.
  - $R_{q'}\left(R_q(p)\right) = R_{q'}(qpq^*) = q'(qpq^*)q'^* = (q'q)p(q'q)^* = R_{q'q}(p)$.

Quaternions and other representations

- **Conversion axis-angle to quaternion**: A vector $v \in R^3$ rotated about a unit axis $a \in R^3$ by an angle $\theta \in \langle 0,2\pi)$ can be computed using quaternions as $R_q(p) = qpq^*$, where $p = (0, v), q = (\cos\theta/2, \sin\theta/2\, a)$.
- **Conversion quaternion to axis-angle**: Let q be a quaternion s.t. $|q| = 1$ expressed in the form $q = (\cos\alpha, \sin\alpha\, a)$, $|a| = 1$. Then q represents rotation about the axis vector a by the angle 2α.
- **Conversion quaternion to rotation matrix**: Let $q = (s, u)$ be a quaternion s.t. $|q| = 1$. Then the vector part of $R_q((0, v))$ is:

$$(s^2 - u \cdot u)v + 2(u \cdot v)u + 2s\, u \times v =$$
$$(s^2 - (1 - s^2))v + 2(u \times (u \times v) + (1 - s^2)v) + 2s\, u \times v =$$
$$v + 2[\![u]\!]^2 v + 2s[\![u]\!]v =$$
$$(I + 2[\![u]\!]^2 + 2s[\![u]\!])v$$

$$|q|^2 = 1^2 = s^2 + u \cdot u$$

$$u \times (u \times v) = (u \cdot v)u - (u \cdot u)v = (u \cdot v)u - (1 - s^2)v$$

Linear interpolation (lerp)

- Given quaternions q, p s.t. $|q| = |p| = 1$ we can linearly interpolate between them by $t \in \langle 0,1 \rangle$:

$$Q(t) = \frac{(1-t)q + tp}{|(1-t)q + tp|}.$$

- Technical issues related to $|(1-t)q + tp|$:
  - Slow – we must compute the square root.
  - The result is not defined when =0.
- **Problem**: The velocity is not constant (increases and decreases) .

  $\Rightarrow$ Visible visual artefact – we prefer **uniform** blending between rotations.
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  - Yes, use **spherical** linear interpolation.

Spherical linear interpolation (slerp)

- Let us first compute a quaternion Δq representing a rotation from a quaternion $q_0 = (s, u)$ to $q_1 = (h, v)$. We assume $|\Delta q| = |q_0| = |q_1| = 1$.

$$\Rightarrow \quad \Delta q q_0 = q_1$$

$$\Delta q = q_1 q_0^{-1} = q_1 q_0^* = (sh + u \cdot v, sv - hu + u \times v) = (\cos\alpha, \sin\alpha\, a).$$

where,

$$\alpha = \cos^{-1}(sh + u \cdot v),\ a = (sv - hu + u \times v)/\sin\alpha \quad (a = 0 \text{ for } \alpha = 0, \pi).$$

- For $t \in \langle 0,1 \rangle$ we define $\Delta q(t) = (\cos t\alpha, \sin t\alpha\, a)$. So, we get:

$$\mathrm{slerp}(q_0, q_1, t) = \Delta q(t) q_0 = (\cos t\alpha, \sin t\alpha\, a) q_0.$$

Quaternion derivative

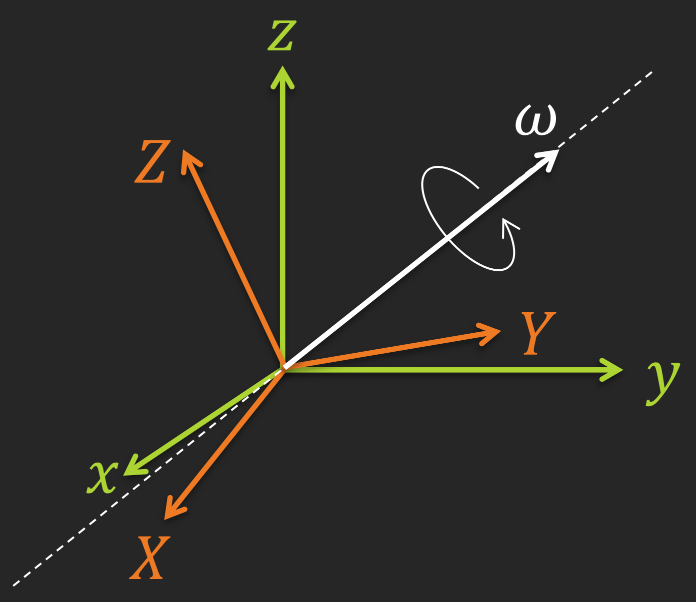
- Let $q(t) = s(t) + u_x(t)\boldsymbol{i} + u_y(t)\boldsymbol{j} + u_z(t)\boldsymbol{k} = (s(t), u(t))$, be a quaternion where $s(t), u_x(t), u_y(t), u_z(t)$ are functions of $t \in R$. Then we define

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \dot{q}(t) = \lim_{\Delta t \to 0} \frac{q(t+\Delta t)-q(t)}{\Delta t} =$$
$$= \lim_{\Delta t \to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t} + \boldsymbol{i}\lim_{\Delta t \to 0} \frac{u_x(t+\Delta t)-u_x(t)}{\Delta t} + \boldsymbol{j}\lim_{\Delta t \to 0} \frac{u_y(t+\Delta t)-u_y(t)}{\Delta t} +$$
$$\boldsymbol{k}\lim_{\Delta t \to 0} \frac{u_z(t+\Delta t)-u_z(t)}{\Delta t} =$$
$$= \frac{\mathrm{d}s(t)}{\mathrm{d}t} + \frac{\mathrm{d}u_x(t)}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}u_y(t)}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}u_z(t)}{\mathrm{d}t}\boldsymbol{k} =$$
$$= \left(\frac{\mathrm{d}s(t)}{\mathrm{d}t}, \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right).$$

- Note: $\int q(t)dt = \int s(t)dt + \boldsymbol{i}\int u_x(t)dt + \boldsymbol{j}\int u_y(t)dt + \boldsymbol{k}\int u_z(t)dt.$

Quaternion derivative

- **Example**: Let's compute a derivative of the orientation of a frame of reference (orange) rotating a constant angular speed $|\omega|$ about the unit axis vector $\widehat{\omega} = \omega/|\omega|$.



- **Solution**:
  - Let $q(t)$ be an orientation (rotation) of the orange frame in the green one (world).
  - We express the rotation about $\widehat{\omega}$:

$$\Delta q(\Delta t) = \left(\cos\frac{|\omega|\Delta t}{2}, \sin\frac{|\omega|\Delta t}{2}\widehat{\omega}\right).$$

  - $\dot{q}(t) = \lim_{\Delta t \to 0}\frac{q(t+\Delta t) - q(t)}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta q(\Delta t)q(t) - q(t)}{\Delta t} = \left(\lim_{\Delta t \to 0}\frac{\Delta q(\Delta t) - (1,0)}{\Delta t}\right)q(t)$

Quaternion derivative

$$= \left(\lim_{\Delta t \to 0} \frac{\cos \frac{|\omega|\Delta t}{2} - 1}{\Delta \mathrm{t}}, \lim_{\Delta t \to 0} \frac{\sin \frac{|\omega|\Delta t}{2}}{\Delta t} \hat{\omega} \right) q(t) \qquad // \cos 2\varphi = 1 - 2\sin^2 \varphi$$

$$= \left(\lim_{\Delta t \to 0} \frac{-2\sin^2 \frac{|\omega|\Delta t}{4}}{\Delta t}, \lim_{\Delta t \to 0} \frac{\sin \frac{|\omega|\Delta t}{2}}{\Delta t} \hat{\omega} \right) q(t)$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(-2\sin^2 \frac{|\omega|t}{4}\right)(0), \frac{\mathrm{d}}{\mathrm{d}t} \left(\sin \frac{|\omega|t}{2}\right)(0) \hat{\omega} \right) q(t)$$

$$= \left(\left(-2 \frac{|\omega|}{4} \cos \frac{|\omega|t}{4} 2\sin \frac{|\omega|t}{4}\right)(0), \left(\frac{|\omega|}{2} \cos \frac{|\omega|t}{2}\right)(0) \hat{\omega} \right) q(t)$$

$$= \left(0, \frac{|\omega|}{2} \hat{\omega} \right) q(t)$$

// Observe: $(0, |\omega|\hat{\omega})$ represents ω.

$$= \frac{1}{2} \omega q(t).$$

Quaternion representation

- We can use quaternions to express any orientation of object in 3D space.

```
public class Orientation {   // Equals to a quaternion q=(s,u), |q|=1.
    float s;                 // the scalar part
    Vector3 u;               // the vector part
};
```

- Pros:
  - Low memory footprint.
  - Fast conversion to rotation matrix (no need to compute cosine & sine).
  - Fast composition of rotations (just multiply the quaternions).
  - We can use lerp and slerp.
- Cons:
  - Less human readable.

What representation to use?

- There is no single winner – each representation has pros and cons.

- Examples:
  - When specifying a rotation along a **coordinate** axis (e.g., world Z), then Euler angles are a good choice.
  - When specifying a rotation along **non-coordinate** axis, then axis-angle representation is a good choice.
  - When composing rotations of some joint of a skeleton, then quaternions can do it quickly.
  - When rotations must be composed with other transformations, then use matrix representation.

- Game engine should provide **all** representations and conversions between them.

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