Solving differential equations

Marek Trtík
PA199

Outline

- Initial value problem for ordinary differential equations.
- Forward Euler's method.
- Backward Euler's method.
- Midpoint method.
- Runge-Kutta methods.

Initial value problem

Initial value problem

- **Initial value problem** (IVP) for the 1st order **ordinary differential equations** (ODE)s:

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}, t), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

  - $\boldsymbol{y}(t) = \left(y_1(t), \dots, y_n(t)\right)^{\mathrm{T}}$ is a vector of **unknown** functions $y_i: \mathbb{R} \to \mathbb{R}$.
  - $\boldsymbol{F}(\boldsymbol{y}, t) = \left(f_1(\boldsymbol{y}(t), t), \dots, f_n(\boldsymbol{y}(t), t)\right)^{\mathrm{T}}$ is a vector of **known** fns $f_i: \mathbb{R}^{n+1} \to \mathbb{R}$.
  - The initial value condition:
    - t_0 **given** time point.
    - $\boldsymbol{y}_0 = \left(y_1(t_0), \dots, y_n(t_0)\right)^{\mathrm{T}}$ is a vector of **known** values of functions y_i at t_0.
- Solution: Any vector of functions $\hat{\boldsymbol{y}}(t) = (\hat{y}_1(t), \dots, \hat{y}_n(t))$ s.t.:

$$\dot{\hat{\boldsymbol{y}}} = \boldsymbol{F}(\hat{\boldsymbol{y}}, t), \qquad \hat{\boldsymbol{y}}(t_0) = \boldsymbol{y}_0$$

- NOTE: We can extend to higher orders, e.g., $\ddot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)$, $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$.
  - We can also have initial condition for derivatives, e.g., $\dot{\boldsymbol{y}}(t_0) = \dot{\boldsymbol{y}}_0$.

Initial value problem

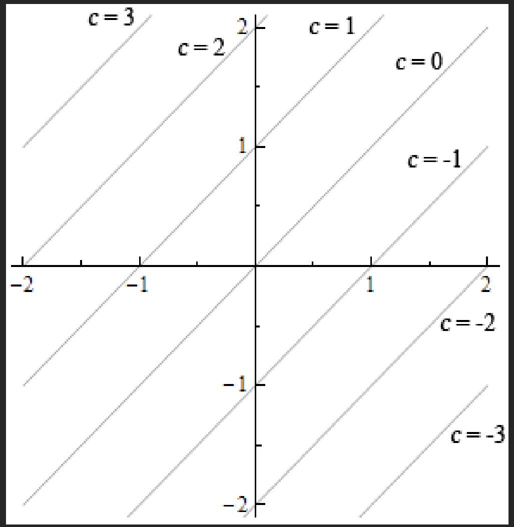
- **Example**: Check that $y(t) = \frac{3}{4} + \frac{c}{t^2}, c \in \mathbb{R}$ is a general solution to $\dot{y} = \frac{3-4y}{2t}$. Find c for which initial condition $y(1) = -4$ is satisfied.
- Solution:
  - $\frac{d}{dt}\left(\frac{3}{4} + \frac{c}{t^2}\right) = -\frac{2c}{t^3}$
  - $\frac{3-4(\frac{3}{4}+\frac{c}{t^2})}{2t} = -\frac{4c}{t^2} \cdot \frac{1}{2t} = -\frac{2c}{t^3}$
  - $\frac{3}{4} + \frac{c}{1^2} = -4 \;\; => c = -\frac{19}{4}$

- In physics simulations:
  - Initial conditions define current state of the system.

Direction field

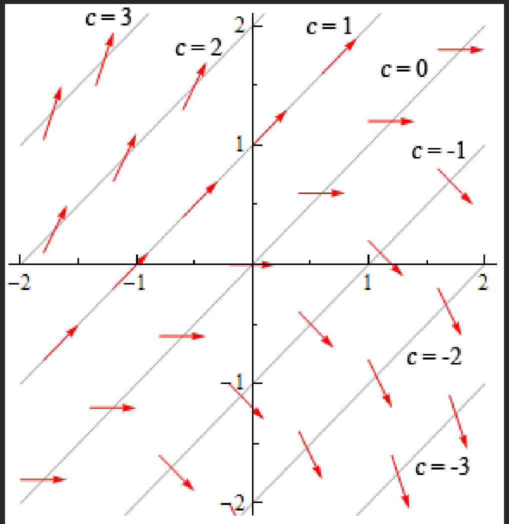
- Plot of a function $\boldsymbol{F(y,t)}$ for some values of $\boldsymbol{y,t}$.
  - Goal: Get visual impression about derivatives $\boldsymbol{y}$.
- **Example**: Show direction field for $\dot{y} = y - t$. [axes: t horizontal, y vertical]

Step 1



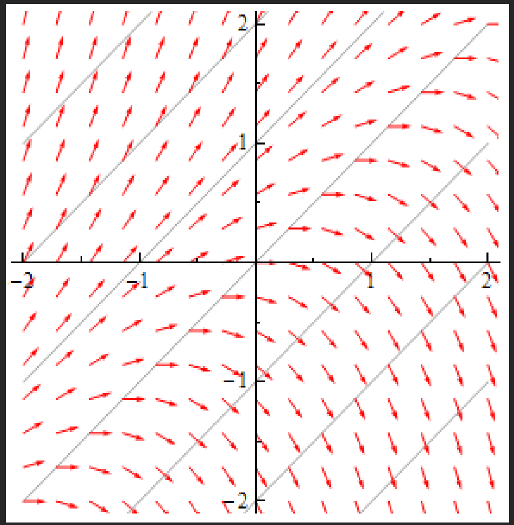
Find contours:
$\dot{y} = y - t = c, c \in \mathrm{R}$.

Step 2



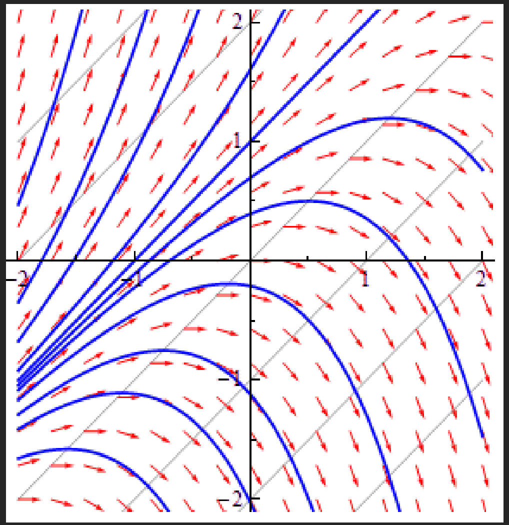
Draw y's slopes using arrows.

Step 3



Draw more arrows.

Step 4



Predict solutions.

Pictures source: [3]

Numerical solution

- The goal is find $\boldsymbol{y}(t_1)$, where $t_1 > t_0$, for a given IVP $\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}, t),\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0$:
  - Start at the initial time t_0 and the initial value $\boldsymbol{y}_0$.
  - Compute a sequence of values $\boldsymbol{y}(t_0 + \Delta t), \boldsymbol{y}(t_0 + 2\Delta t), \dots, \boldsymbol{y}(t_0 + n\Delta t)$, where $t_1 = t_0 + n\Delta t$.

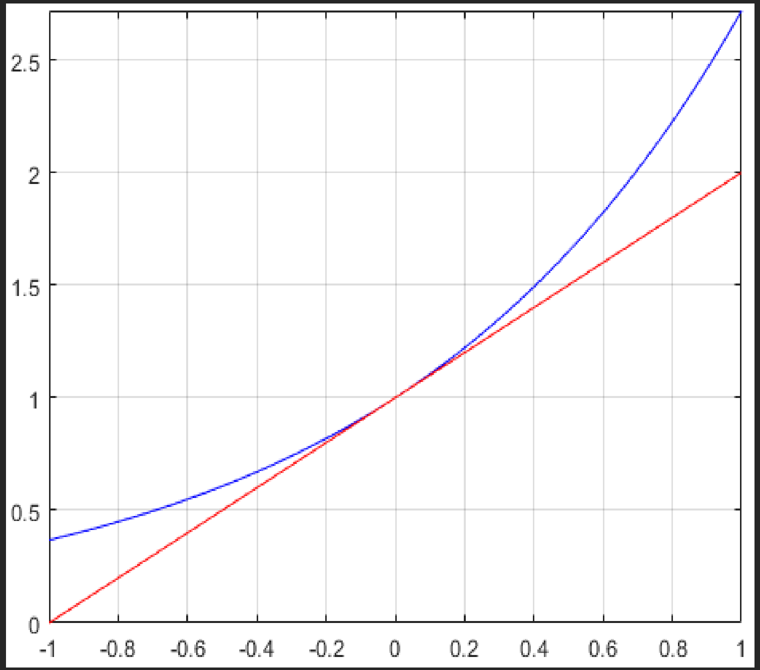
- There are two kinds of methods:
  - **Explicit** methods:
    - Compute $\boldsymbol{y}(t_0 + \Delta t)$ by a function $\mathcal{F}(\boldsymbol{F}, \boldsymbol{y}_0, t_0, \Delta t)$ of **current** state of the system.
  - **Implicit** methods:
    - Compute $\boldsymbol{y}(t_0 + \Delta t)$ by a solution of an equation $\mathcal{F}\left(\boldsymbol{F}, \boldsymbol{y}_0, t_0, \Delta t, \boldsymbol{y}(t_0 + \Delta t)\right) = 0$ over the **current and future** state of the system.

Taylor theorem

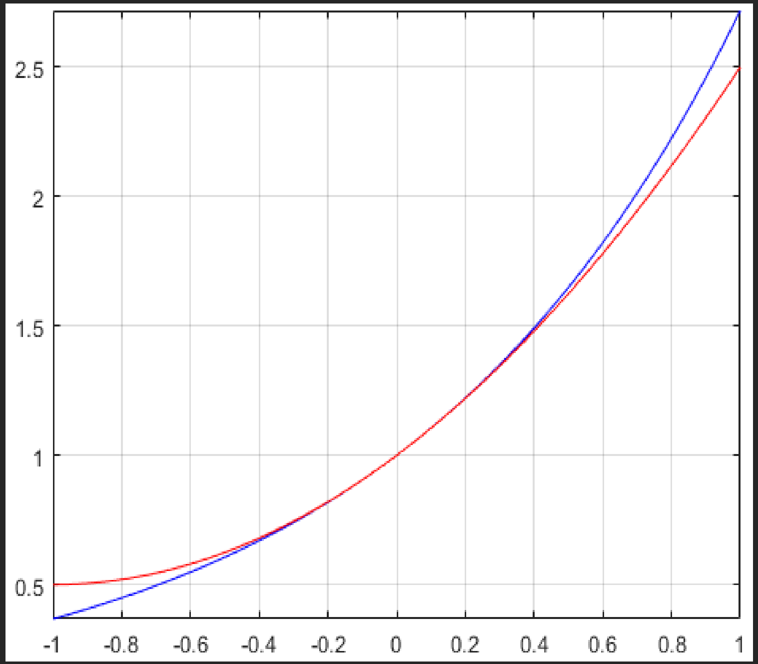
- For a k-times differentiable function $y: \mathbb{R} \to \mathbb{R}$ at a point $t_0 \in D(y)$ there exists a polynomial $P_k: \mathbb{R} \to \mathbb{R}$ and a functions $R: \mathbb{R} \to \mathbb{R}$ s.t.
  - $y(t_0 + \Delta t) = P_k(t_0 + \Delta t) + \Delta t^k R(t_0 + \Delta t)$
  - $P_k(t_0 + \Delta t) = y(t_0) + \Delta t \dot{y}(t_0) + \frac{\Delta t^2}{2!} \ddot{y}(t_0) + \cdots + \frac{\Delta t^k}{k!} y^{(k)}(t_0)$
  - $\lim_{\Delta t \to 0} R(t_0 + \Delta t) = 0$

Numerical solution

- **Examples** (Taylor approximation):



$y(t) = e^t$, $P_1(t) = 1 + t$, $t_0 = 0$.



$y(t) = e^t$, $P_2(t) = 1 + t + \frac{t^2}{2}$, $t_0 = 0$.

"$\mathcal{O}$" error notation

- What is the error from the approximation using P_k:
$$y(t_0 + \Delta t) \approx P_k(t_0 + \Delta t)$$
- It is a distance from the exact value $P_{k+1}(t_0 + \Delta t) + \Delta t^{k+1} R(t_0 + \Delta t)$:
$$\text{error} = P_{k+1}(t_0 + \Delta t) + \Delta t^{k+1} R(t_0 + \Delta t) - P_k(t_0 + \Delta t)$$
$$= \frac{\Delta t^{k+1}}{(k+1)!} y^{(k+1)}(t_0) + \Delta t^{k+1} R(t_0 + \Delta t)$$
- For small Δt the error is proportional to the term Δt^{k+1}. Therefore,

$$\boxed{y(t_0 + \Delta t) = P_k(t_0 + \Delta t) + \mathcal{O}(\Delta t^{k+1})}$$

Forward Euler's method

- We get forward Euler's method, when we approximate y by P_1 at t_0:

$$\begin{aligned} y(t_0 + \Delta t) &\approx y(t_0) + \Delta t \dot{y}(t_0) \\ &= y(t_0) + \Delta t F(y(t_0), t_0) \end{aligned}$$

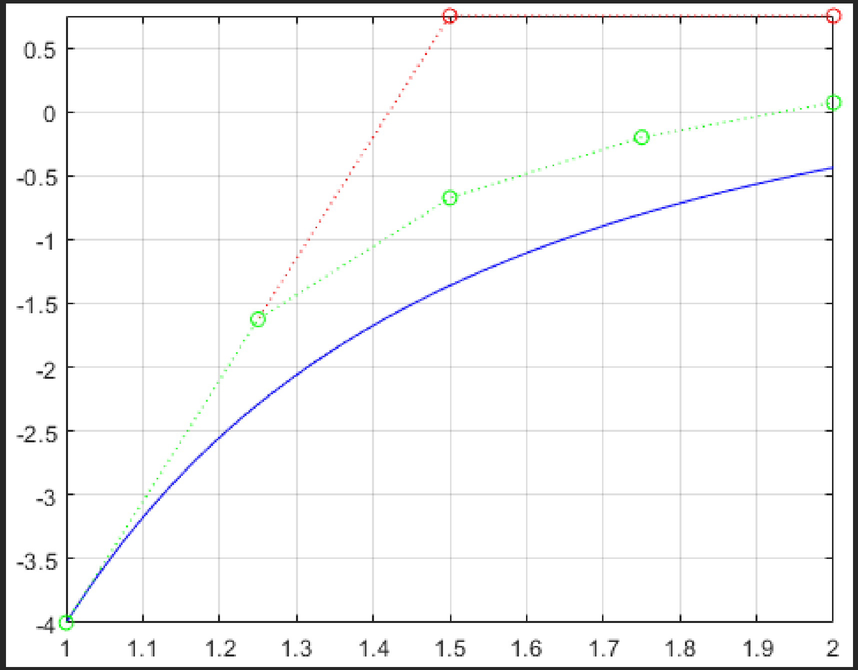
- We see that forward Euler's method is an **explicit** method.

Forward Euler's method

- **Example**: Let IPV be $\dot{y} = \frac{3-4y}{2t}, y(1) = -4$. Compute $y(2)$ by forward Euler's method. [Note: Exact solution is $y(t) = \frac{3}{4} - \frac{19}{4t^2}$]
- Solution:
  - Let's choose a time step $\Delta t = \frac{1}{2}$ => We must apply the method 2 times.
  - $y(1) = -4$ ← from the initial condition.
  - $y\left(\frac{3}{2}\right) = y\left(1 + \frac{1}{2}\right) \approx y(1) + \frac{1}{2}F(y(1), 1) = -4 + \frac{1}{2} \cdot \frac{3-4(-4)}{2 \cdot 1} = \frac{3}{4}$ ← 1st iteration
  - $y(2) = \frac{3}{4} + \frac{1}{2} \cdot \frac{3-4 \cdot \frac{3}{4}}{2 \cdot \frac{3}{2}} = \frac{3}{4}$ ← 2nd iteration
- We see the method is **simple** and **fast**.

Forward Euler's method

- Low accuracy issue:
  - $\mathcal{O}(\Delta t^2)$ error in each iteration.
- **Example**:



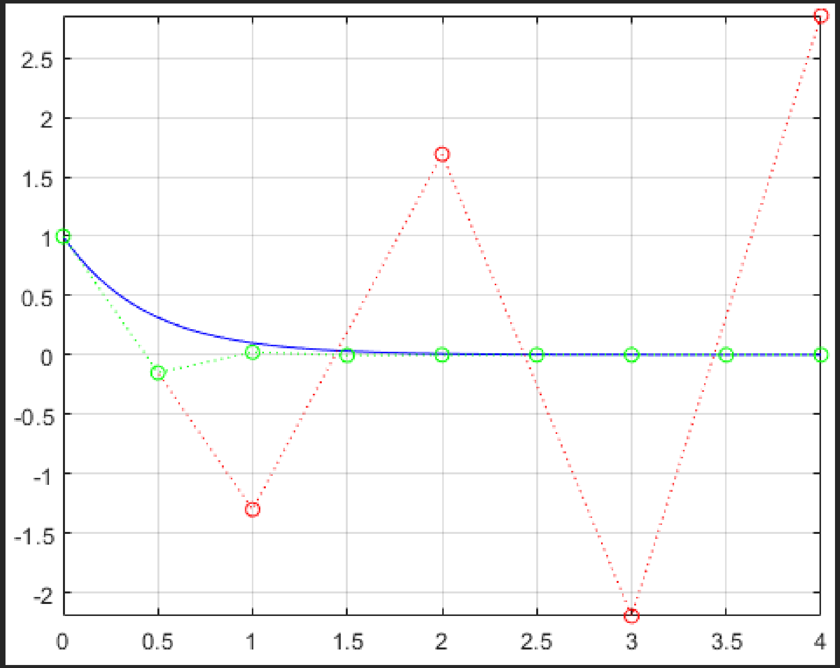
IVP: $\dot{y} = \frac{3-4y}{2t}, y(1) = -4.$

Euler's method: $\Delta t = \frac{1}{2}, \Delta t = \frac{1}{4}.$

Exact solution: $y(t) = \frac{3}{4} - \frac{19}{4t^2}$

Forward Euler's method

- Instability issue:
  - The iteration process may diverge.
- **Example**:



IVP: $\dot{y} = -2.3y, y(0) = 1.$

Euler's method: $\Delta t = 1, \Delta t = \frac{1}{2}.$

Exact solution: $y(t) = e^{-2.3t}.$

Forward Euler's method

- What can we do with the issues?
  - Use smaller time step Δt to reduce the error and/or avoid the instability.
  - But we then need more iterations => slower simulation.
  - Choose more accurate/stable solver.

- Suggestion for seminar: Implement method "ODE_Euler_forward".

```
void ODE _Euler_forward(
        std::vector<float> const& y0,        // x, v of particle(s)
        std::vector<F_y_t> const& Fyt,       // ẋ, v̇ of particle(s), i.e. v, F/m
        float& t,                            // current time (to be updated)
        float const dt,                      // time step
        std::vector<float>& y)               // integrated x, v of particle(s)
{ TODO }
```

Forward Euler's method

- **Example**: Let's consider a particle $\mathcal{P}(t) = (\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{F}, m)$, where $m = 0.1\text{kg}$, in a homogenous gravity field with $\boldsymbol{g} = (0,0,-10)^\top \text{m} \cdot \text{s}^{-2}$. At time $t = 1$s we have $\boldsymbol{x} = (1,-1,5)^\top \text{m}, \boldsymbol{v} = (1,0,0)^\top \text{m} \cdot \text{s}^{-1}$. Using forward Euler's method with $\Delta t = 0.5$s compute $\mathcal{P}(2)$.
- Solution: Particle moves by Newton's equations of motion:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \qquad \dot{\boldsymbol{v}}(t) = \frac{\boldsymbol{F}}{m}$$

Therefore: $\boldsymbol{x}(1.5) = \begin{pmatrix} 1 + 0.5 \cdot 1 \\ -1 + 0.5 \cdot 0 \\ 5 + 0.5 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1 \\ 5 \end{pmatrix}, \boldsymbol{v}(1.5) = \begin{pmatrix} 1 + 0.5 \frac{0.1 \cdot 0}{0.1} \\ 0 + 0.5 \frac{0.1 \cdot 0}{0.1} \\ 0 + 0.5 \frac{0.1 \cdot -10}{0.1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}.$

$$\boldsymbol{x}(2) = (2,-1,0)^\top,\ \boldsymbol{v}(2) = (1,0,-10)^\top.$$

Backward Euler’s method

Backward Euler's method

- From the fundamental theorem of the calculus:

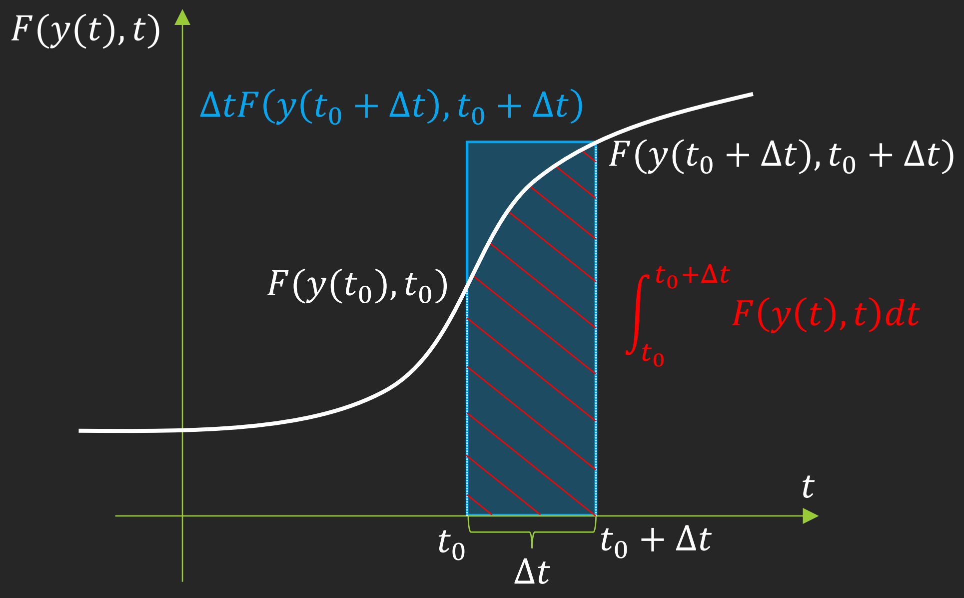
$$\int_{t_0}^{t_0+\Delta t} \dot{y}(t)dt = y(t_0+\Delta t) - y(t_0).$$

- Therefore,

$$y(t_0+\Delta t) = y(t_0) + \int_{t_0}^{t_0+\Delta t} F(y(t),t)dt\,.$$

- We can approximate the integral by "right-hand" rectangle:

$$\int_{t_0}^{t_0+\Delta t} F(y(t),t)dt \approx \Delta t F(y(t_0+\Delta t), t_0+\Delta t).$$



Backward Euler's method

- Backward Euler's method leads to this equation:

$$y(t_0 + \Delta t) \approx y(t_0) + \Delta t F(y(t_0 + \Delta t), t_0 + \Delta t)$$

- Backward Euler's method is an **implicit** method.
- We must solve this equation to obtain the unknown $y(t_0 + \Delta t)$:

$$y(t_0 + \Delta t) - y(t_0) - \Delta t F(y(t_0 + \Delta t), t_0 + \Delta t) = 0$$

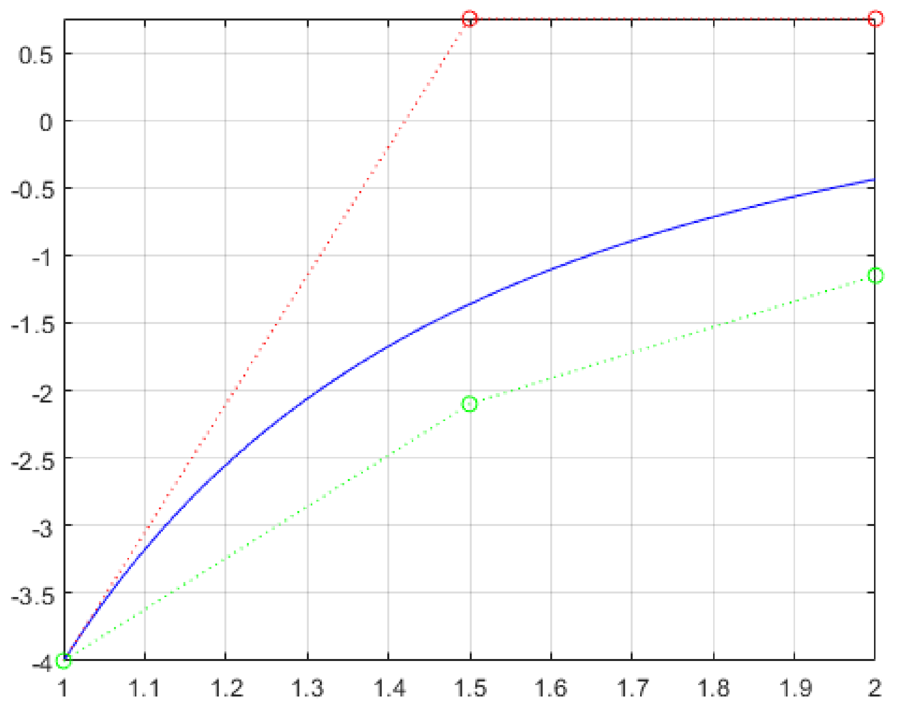
  - Use any available method for solving the equation, e.g., Newton's method ($y^{[k+1]} = y^{[k]} - \mathcal{F}(y^{[k]})/\dot{\mathcal{F}}(y^{[k]})$).
- Note: If we have system of ODEs, then we get system of equations.

Backward Euler's method

- **Example**: Let IVP be $\dot{y} = \frac{3-4y}{2t}, y(1) = -4$. Compute $y(2)$ by backward Euler's method. [Note: Exact solution is $y(t) = \frac{3}{4} - \frac{19}{4t^2}$]
- Solution:
  - Let's choose a time step $\Delta t = \frac{1}{2}$ => We must apply the method 2 times.
  - $y(1) = -4$ ← from the initial condition.
  - $y\left(\frac{3}{2}\right) = y\left(1 + \frac{1}{2}\right) = y(1) + \frac{1}{2} F\left(y\left(\frac{3}{2}\right), \frac{3}{2}\right) = -4 + \frac{1}{2} \cdot \frac{3-4y\left(\frac{3}{2}\right)}{2 \cdot \frac{3}{2}} = -\frac{7}{2} - \frac{2}{3} y\left(\frac{3}{2}\right)$
    - We solve: $y\left(\frac{3}{2}\right) = -\frac{7}{2} - \frac{2}{3} y\left(\frac{3}{2}\right)$ => $y\left(\frac{3}{2}\right) = -\frac{7}{2\left(1+\frac{2}{3}\right)} = -\frac{21}{10}$ ← 1st iteration
  - $y(2) = -\frac{21}{10} + \frac{1}{2} \cdot \frac{3-4y(2)}{2 \cdot 2} = -\frac{21}{10} + \frac{3}{8} - \frac{1}{2} y(2)$ => $y(2) = -\frac{23}{20}$ ← 2nd iteration

Backward Euler's method

- We can plot out result and compare it with forward Euler's method:



IVP: $\dot{y} = \frac{3-4y}{2t}, y(1) = -4.$

Backward Euler: $\Delta t = \frac{1}{2}.$

Forward Euler: $\Delta t = \frac{1}{2}.$

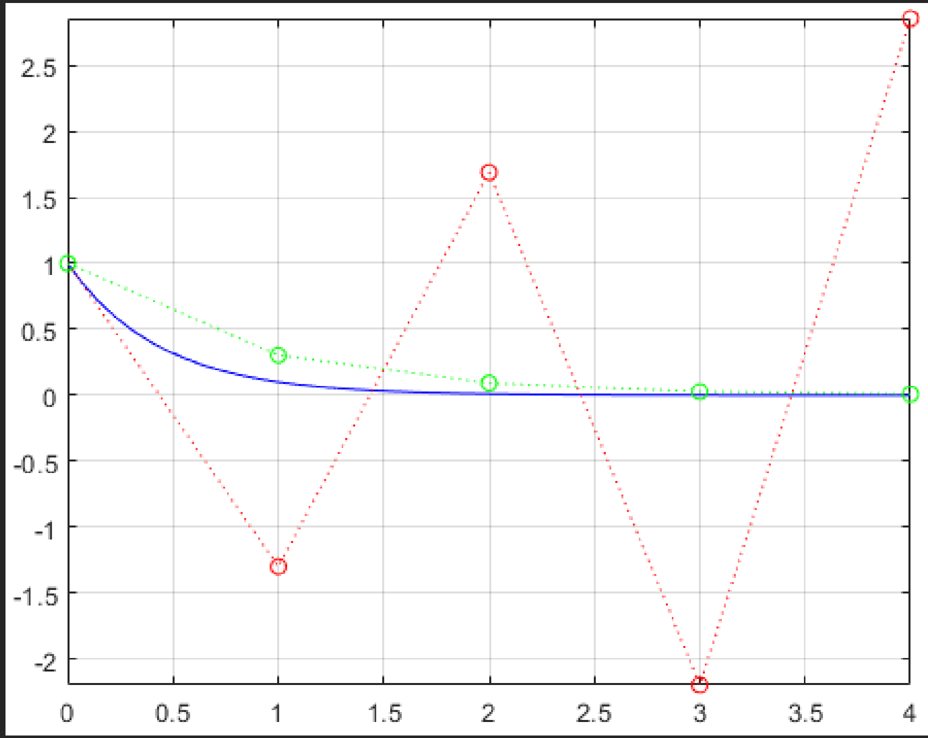
Exact solution: $y(t) = \frac{3}{4} - \frac{19}{4t^2}$

Backward Euler's method

- **Example 2**: Let IPV be $\dot{y} = -2.3y, y(0) = 1$. Compute $y(4)$ by backward Euler's method.
- Solution:
  - Let's choose a time step $\Delta t = 1$ => We must apply the method 4 times.
  - $y(0) = 1$ ← from the initial condition.
  - $y(1) = 1 + 1 \cdot -2.3y(1)$ => $y(1) = \frac{1}{1+2.3} = \frac{10}{33}$ ← 1st iteration
  - $y(2) = \frac{10}{33} + 1 \cdot -2.3y(2)$ => $y(2) = \frac{10}{33} \cdot \frac{10}{33} = \left(\frac{10}{33}\right)^2$ ← 2nd iteration
  - $y(3) = \left(\frac{10}{33}\right)^2 - 2.3y(3)$ => $y(3) = \left(\frac{10}{33}\right)^2 \cdot \frac{10}{33} = \left(\frac{10}{33}\right)^3$ ← 3rd iteration
  - $y(4) = \left(\frac{10}{33}\right)^3 - 2.3y(4)$ => $y(4) = \left(\frac{10}{33}\right)^4$ ← 4th iteration
  - Note: Observe the geometric progression $y_{k+1} = qy_k, q = \frac{10}{33}$ => $y_k = q^k y_0$.

Backward Euler's method

- We can plot out result and compare it with forward Euler's method:



IVP: $\dot{y} = -2.3y, y(0) = 1.$
Backward Euler: $\Delta t = 1.$
Forward Euler: $\Delta t = 1.$
Exact solution: $y(t) = e^{-2.3t}$

Backward Euler's method

- Properties of backward Euler's method
  - Hard to implement.
  - Requires solving an equation or a system of equations.
  - $\mathcal{O}(\Delta t^2)$ error in each iteration.
  - Stable for large time step Δt.
- Choice between forward/backward Euler's method depends on a problem. "Rule of thumb":
  - Prefer forward method for "stable" problems.
  - Prefer backward method for "stiff" problems.

Midpoint method

Midpoint method

- Let's try to approximate y by P_2 at t_0:

$$y(t_0+\Delta t) = y(t_0) + \Delta t\dot{y}(t_0) + \frac{\Delta t^2}{2!}\ddot{y}(t_0) + \mathcal{O}(\Delta t^3)$$

$$= y(t_0) + \Delta t F(y(t_0), t_0) + \underbrace{\frac{\Delta t^2}{2}\dot{F}(y(t_0), t_0) + \mathcal{O}(\Delta t^3)}_{(*)}$$

- How to compute $\dot{F}$?
  - Using the chain rule, we get: $\dot{F} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y}\dot{y} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y}F$
  - Not much better, because we still do not know $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial y}$.
- So, let's try to approximate F using P_1 …
  - Note: We must use a 2-variables version of Taylor's theorem.

Midpoint method

$$F(y(t_0)+\Delta y, t_0+\Delta t) = F(y(t_0),t_0) + \Delta y \frac{\partial F}{\partial y}(y(t_0),t_0) + \Delta t \frac{\partial F}{\partial t}(y(t_0),t_0) + \mathcal{O}(\Delta y^2 + \Delta t^2)$$

- Let's substitute: $\Delta y \to \frac{\Delta t}{2} F(y(t_0),t_0), \quad \Delta t \to \frac{\Delta t}{2}$

$$F\left(y(t_0) + \frac{\Delta t}{2}F(y(t_0),t_0), t_0 + \frac{\Delta t}{2}\right) = F(y(t_0),t_0) + \underbrace{\frac{\Delta t}{2}F(y(t_0),t_0)\frac{\partial F}{\partial y}(y(t_0),t_0) + \frac{\Delta t}{2}\frac{\partial F}{\partial t}(y(t_0),t_0)}_{\frac{\Delta t}{2}\dot{F}(y(t_0),t_0)} + \mathcal{O}(\Delta t^2)$$

$$F\left(y(t_0) + \frac{\Delta t}{2}F(y(t_0),t_0), t_0 + \frac{\Delta t}{2}\right) = F(y(t_0),t_0) + \frac{\Delta t}{2}\dot{F}(y(t_0),t_0) + \mathcal{O}(\Delta t^2)$$

$$\frac{\Delta t}{2}\dot{F}(y(t_0),t_0) + \mathcal{O}(\Delta t^2) = F\left(y(t_0) + \frac{\Delta t}{2}F(y(t_0),t_0), t_0 + \frac{\Delta t}{2}\right) - F(y(t_0),t_0)$$

$$\underbrace{\frac{\Delta t^2}{2}\dot{F}(y(t_0),t_0) + \mathcal{O}(\Delta t^3)}_{(*)} = \Delta t F\left(y(t_0) + \frac{\Delta t}{2}F(y(t_0),t_0), t_0 + \frac{\Delta t}{2}\right) - \Delta t F(y(t_0),t_0)$$

Midpoint method

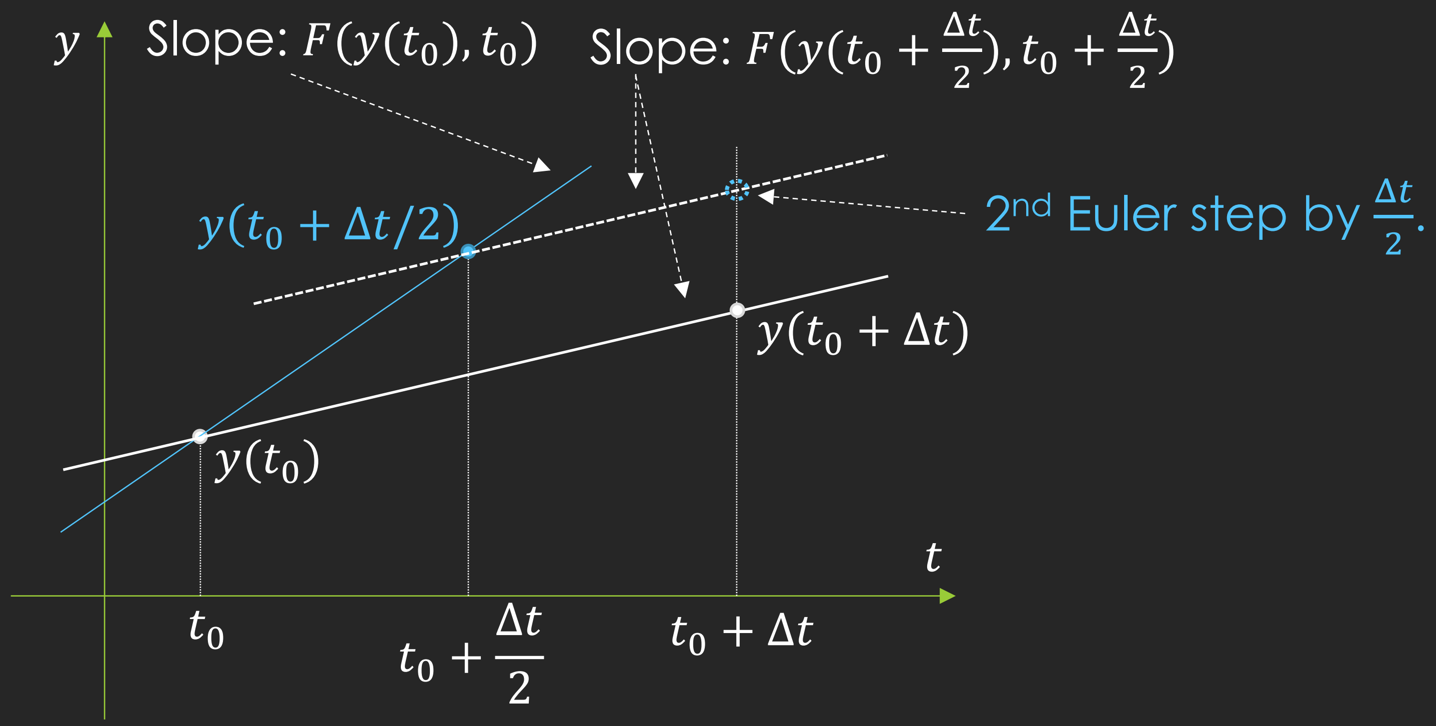
$$y(t_0 + \Delta t) = y(t_0) + \Delta t F(y(t_0), t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right) - \Delta t F(y(t_0), t_0)$$

$$\boxed{y(t_0 + \Delta t) = y(t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right)}$$

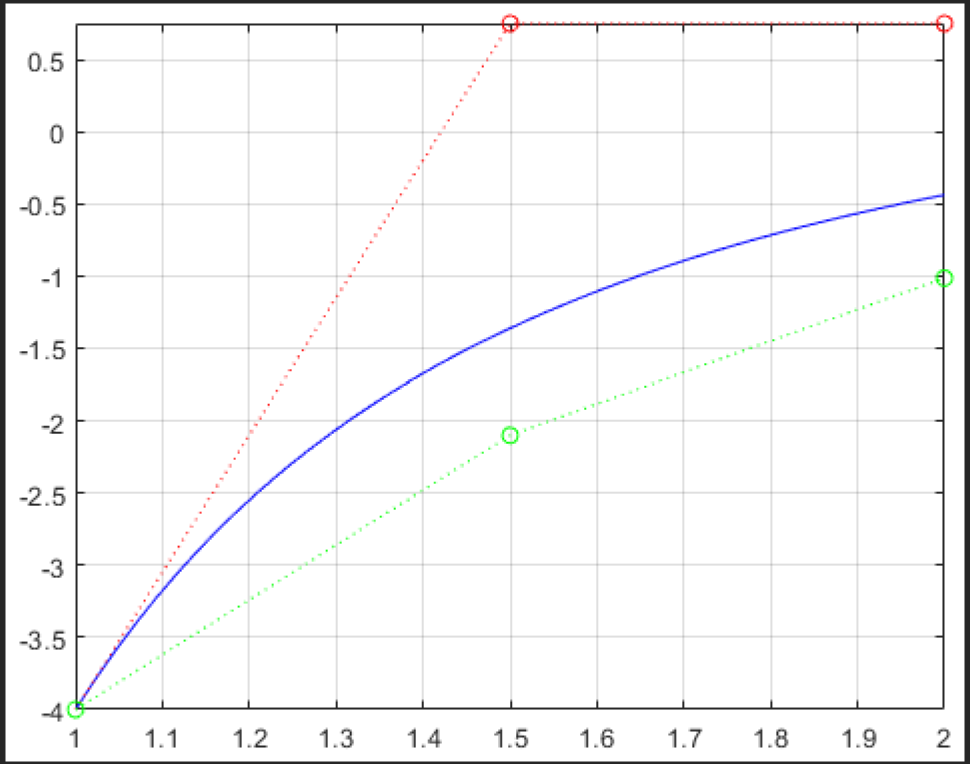
- This is an **explicit** method.
- This method is more accurate than Euler's method:
  - Euler: $\mathcal{O}(\Delta t^2)$
  - Midpoint: $\mathcal{O}(\Delta t^3)$

Midpoint method

$$y(t_0 + \Delta t) = y(t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right)$$



Midpoint method



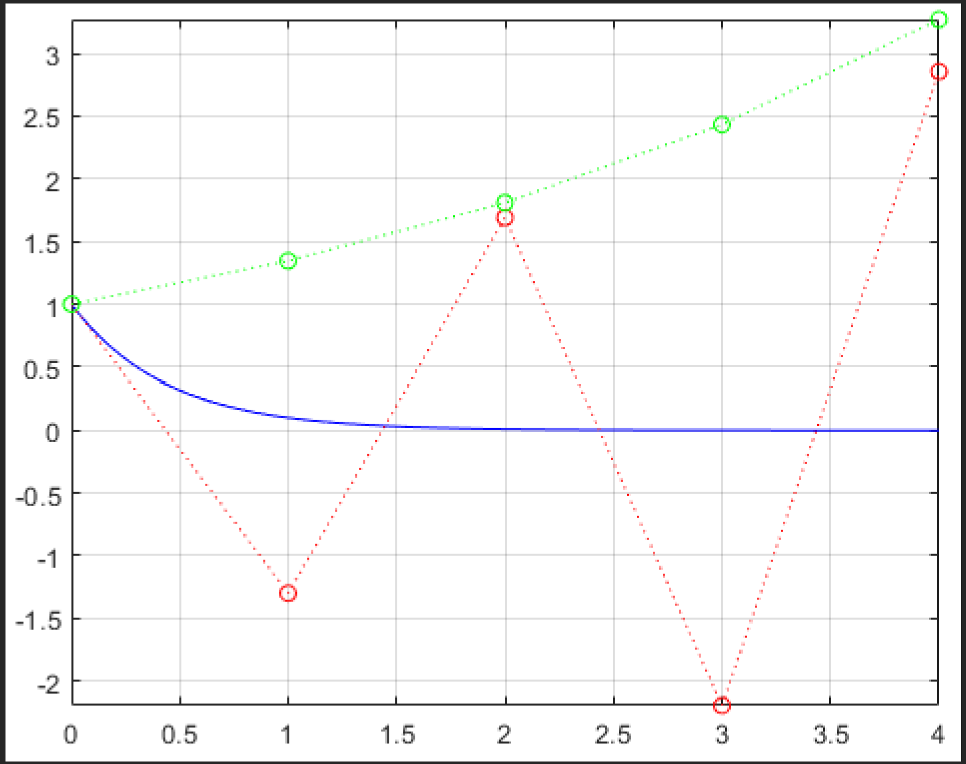
IVP: $\dot{y} = \frac{3-4y}{2t}, y(1) = -4.$

Midpoint method: $\Delta t = \frac{1}{2}.$

Forward Euler: $\Delta t = \frac{1}{2}.$

Exact solution: $y(t) = \frac{3}{4} - \frac{19}{4t^2}$

Midpoint method



IVP: $\dot{y} = -2.3y, y(0) = 1.$
Midpoint method: $\Delta t = 1.$
Forward Euler: $\Delta t = 1.$
Exact solution: $y(t) = e^{-2.3t}$

Midpoint method

- There is also **implicit** version of the midpoint method.
- From the fundamental theorem of the calculus:

$$y(t_0 + \Delta t) = y(t_0) + \int_{t_0}^{t_0+\Delta t} F(y(t), t)dt\,.$$

- We can approximate the integral by "midpoint" rectangle:

$$\int_{t_0}^{t_0+\Delta t} F(y(t), t)dt \approx \Delta t F\left(\frac{y(t_0) + y(t_0 + \Delta t)}{2}, \frac{t_0 + (t_0 + \Delta t)}{2}\right)$$

- Therefore, we get

$$\boxed{y(t_0 + \Delta t) = y(t_0) + \Delta t F\left(\frac{y(t_0) + y(t_0 + \Delta t)}{2}, t_0 + \frac{\Delta t}{2}\right)}$$

Runge-Kutta methods

Runge-Kutta methods

- In general, we can approximate the integral as follows:

$$\int_{t_0}^{t_0+\Delta t} F(y(t),t)dt \approx \Delta t \sum_{i=1}^{n} b_i F(y(t_0 + c_i \Delta t), t_0 + c_i \Delta t)$$

- The problem is that values $y(t_0 + c_i \Delta t)$ are **unknown!**
- Runge-Kutta methods solve the issue by this **substitution**:

$$k_1 = F(y(t_0), t_0)$$

$$k_i = F\left(y(t_0) + \Delta t \sum_{j=1}^{i-1} a_{i,j} k_j \, , t_0 + c_i \Delta t \right), \qquad \text{s.t.} \sum_{j=1}^{i-1} a_{i,j} = c_i$$

$$\int_{t_0}^{t_0+\Delta t} F(y(t),t)dt \approx \Delta t \sum_{i=1}^{n} b_i k_i$$

Runge-Kutta methods

- Therefore, Runge-Kutta of order n is defined as:

$$y(t_0 + \Delta t) = y(t_0) + \Delta t \sum_{i=1}^{n} b_i k_i \,,$$

where terms k_i were defined on the previous slide.

- However, we must **compute** the numbers $a_{i,j}, b_i, c_i$ so that resulting expression yields an approximation by **Taylor's polynomial** P_n.

Runge-Kutta methods

- **Example**: Runge-Kutta method of order 1 (i.e. $n = 1$):

$$k_1 = F(y(t_0), t_0)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t b_1 k_1 = y(t_0) + \Delta t b_1 F(y(t_0), t_0)$$

What value we should choose for b_1? We compare $y(t_0 + \Delta t)$ with P_1.

$$P_1(t_0 + \Delta t) = y(t_0) + \Delta t \dot{y}(t_0) = y(t_0) + \Delta t F(y(t_0), t_0)$$

Therefore, b_1 must be 1.

- Observation: Euler's method is Runge-Kutta method of order 1.

Runge-Kutta methods

- **Example**: Runge-Kutta method of order 2 (i.e. $n = 2$):

$$k_1 = F(y(t_0), t_0)$$
$$k_2 = F\big(y(t_0) + \Delta t a_{2,1} k_1, t_0 + c_2 \Delta t\big)$$
$$y(t_0 + \Delta t) = y(t_0) + \Delta t b_1 k_1 + \Delta t b_2 k_2$$

We compute $a_{2,1}, b_1, b_2$ by comparison of $y(t_0 + \Delta t)$ with P_2.

$$P_2(t_0 + \Delta t) = y(t_0) + \Delta t \dot{y}(t_0) + \frac{\Delta t^2}{2!} \ddot{y}(t_0)$$
$$= y(t_0) + \Delta t F(y(t_0), t_0) + \frac{\Delta t^2}{2} \dot{F}(y(t_0), t_0)$$
$$= y(t_0) + \Delta t F(y(t_0), t_0) + \frac{\Delta t^2}{2} \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} F \right) (y(t_0), t_0)$$

Runge-Kutta methods

For the comparison let's approximate k_2 by P_1:

$$k_2 = F\left(y(t_0) + \Delta t a_{2,1} k_1, t_0 + c_2 \Delta t\right)$$
$$\approx F(y(t_0), t_0) + \Delta t(c_2 \frac{\partial F}{\partial t} + a_{2,1} k_1 \frac{\partial F}{\partial y})(y(t_0), t_0)$$
$$= F(y(t_0), t_0) + \Delta t(c_2 \frac{\partial F}{\partial t} + a_{2,1} F \frac{\partial F}{\partial y})(y(t_0), t_0)$$

When we substitute the approximated k_2 we get:

$$y(t_0 + \Delta t) = y(t_0) + \Delta t b_1 F(y(t_0), t_0)$$
$$+ \Delta t b_2 (F(y(t_0), t_0) + \Delta t(c_2 \frac{\partial F}{\partial t} + a_{2,1} F \frac{\partial F}{\partial y})(y(t_0), t_0))$$
$$= y(t_0) + \Delta t(b_1 + b_2) F(y(t_0), t_0)$$
$$+ \Delta t^2 b_2 \left(c_2 \frac{\partial F}{\partial t} + a_{2,1} \frac{\partial F}{\partial y} F\right)(y(t_0), t_0)$$

Runge-Kutta methods

So, we must solve this system of equations:

$$b_1 + b_2 = 1, \qquad b_2 c_2 = \frac{1}{2}, \qquad b_2 a_{2,1} = \frac{1}{2}.$$

One possible solution is: $b_1 = 0, b_2 = 1, c_2 = \frac{1}{2}, a_{2,1} = \frac{1}{2}$.

(Note: Another solution is: $b_1 = \frac{1}{2}, b_2 = \frac{1}{2}, c_2 = 1, a_{2,1} = 1$)

We get the result:

$$k_1 = F(y(t_0), t_0)$$

$$k_2 = F\left(y(t_0) + \frac{\Delta t}{2} k_1, t_0 + \frac{\Delta t}{2}\right)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t k_2 = y(t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right).$$

- Observation: Midpoint method is Runge-Kutta method of order 2.

Runge-Kutta methods

- **Example**: Runge-Kutta method of order 4:

$$k_1 = F(y(t_0), t_0)$$

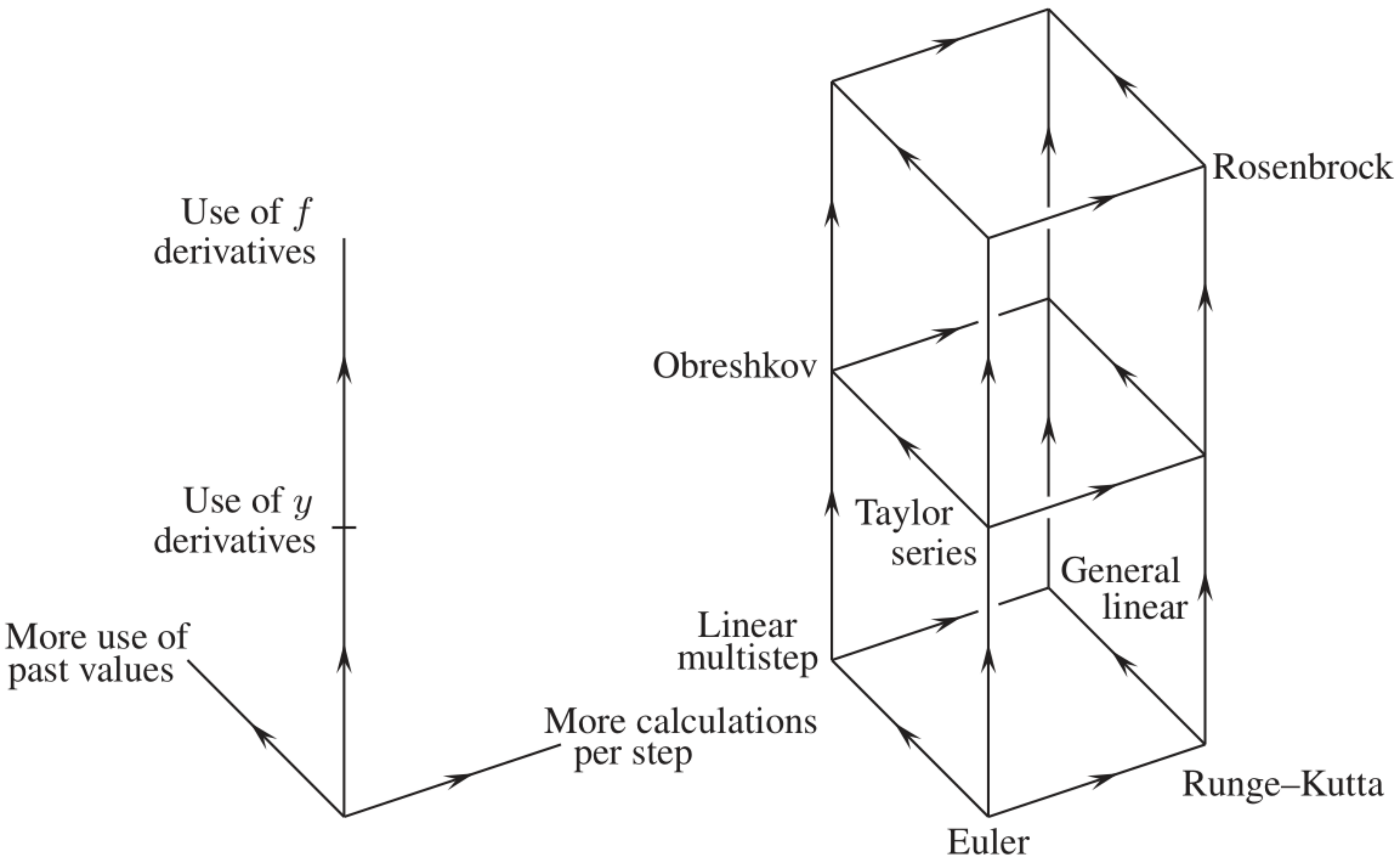
$$k_2 = F\left(y(t_0) + \frac{k_1}{2}, t_0 + \frac{\Delta t}{2}\right)$$

$$k_3 = F\left(y(t_0) + \frac{k_2}{2}, t_0 + \frac{\Delta t}{2}\right)$$

$$k_4 = F(y(t_0) + k_3, t_0 + \Delta t)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t\left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right)$$

Schema of numerical methods



Picture source: [2]

References

[1] A. Witkin, D. Baraff; *Differential Equation Basics*; Physically Based Modeling: Principles and Practice, 1997

[2] J.C.Butcher; Numerical methods for ordinary differential equations; 3rd edition, Wiley, 2016.

[3] https://tutorial.math.lamar.edu/Classes/DE/DE.aspx