

Solving differential equations

Marek Trtík
PA199

Outline

- ▶ Initial value problem for ordinary differential equations.
- ▶ Forward Euler's method.
- ▶ Backward Euler's method.
- ▶ Midpoint method.
- ▶ Runge-Kutta methods.

Initial value problem

Initial value problem

- ▶ Initial value problem (IVP) for the n^{th} order ordinary differential equations (ODE)s:

$$\dot{y} = F(y, t), \quad y(t_0) = y_0$$

- ▶ $y(t) = (y_1(t), \dots, y_n(t))^T$ is a vector of unknown functions $y_i: \mathbb{R} \rightarrow \mathbb{R}$.
- ▶ $F(y, t) = (f_1(y(t), t), \dots, f_n(y(t), t))^T$ is a vector of known fns $f_i: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$.
- ▶ The initial value condition:
 - ▶ t_0 given time point.
 - ▶ $y_0 = (y_1(t_0), \dots, y_n(t_0))^T$ is a vector of known values of functions y_i at t_0 .
- ▶ Solution: Any vector of functions $\hat{y}(t) = (\hat{y}_1(t), \dots, \hat{y}_n(t))$ s.t.
$$\dot{\hat{y}} = F(\hat{y}, t), \quad \hat{y}(t_0) = y_0$$
- ▶ NOTE: We can extend to higher orders, e.g. $\dot{y} = F(y, \dot{y}, t)$, $y(t_0) = y_0$.
- ▶ We can also have initial condition for derivatives, e.g. $\dot{y}(t_0) = \dot{y}_0$.

Initial value problem

▶ Example: Check that $y(t) = \frac{3}{2} + \frac{c}{t}$, $c \in \mathbb{R}$ is a general solution to $y' = \frac{3-2y}{2t}$. Find c for which initial condition $y(1) = -4$ is satisfied.

▶ Solution:

$$\frac{d}{dt} \left(\frac{3}{2} + \frac{c}{t} \right) = -\frac{2c}{t^2}$$

$$\frac{3-4\left(\frac{3}{2} + \frac{c}{t}\right)}{2t} = \frac{4c-1}{t^2} = -\frac{2c}{t^2}$$

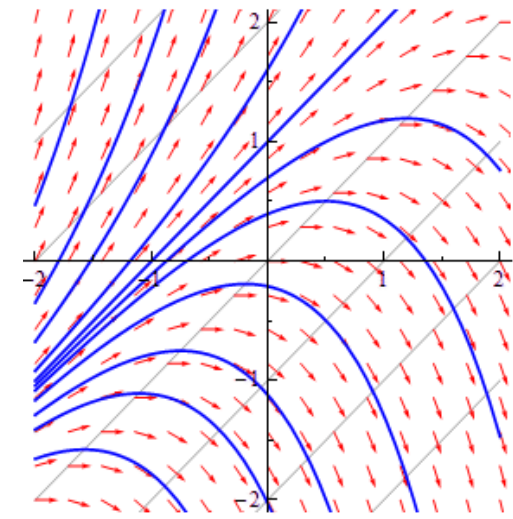
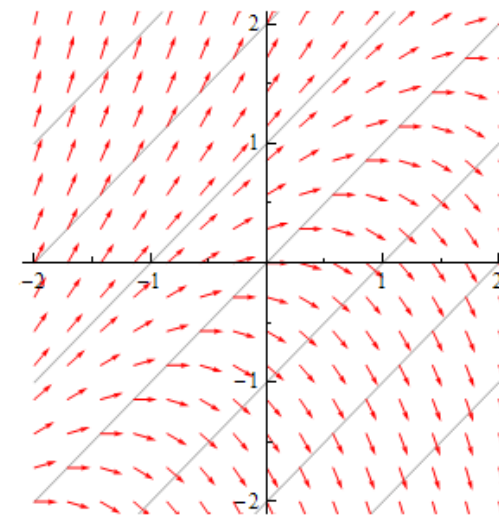
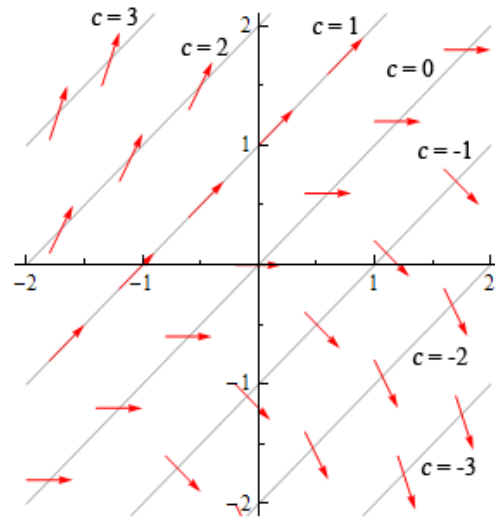
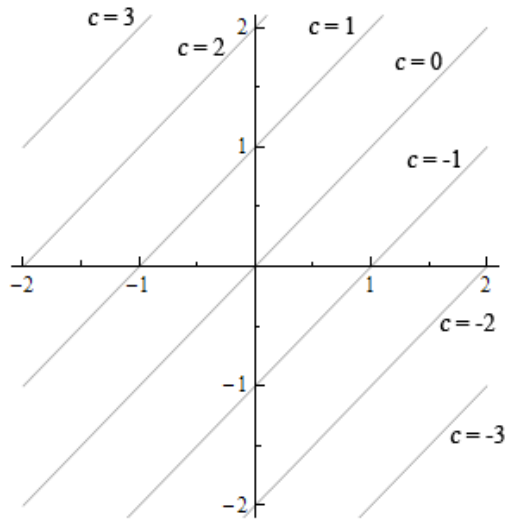
$$\frac{3}{2} + \frac{c}{t} = -4 \Rightarrow c = -\frac{19}{4}$$

▶ In physics simulations:

▶ Initial conditions define current state of the system.

Direction field

- ▶ You are given y' by or some values of x, y .
- ▶ Goal: Get your impression about derivatives.
- ▶ Example: Show direction field of $y' = y - 2$. axes, horizontal, vertical



Numerical solution

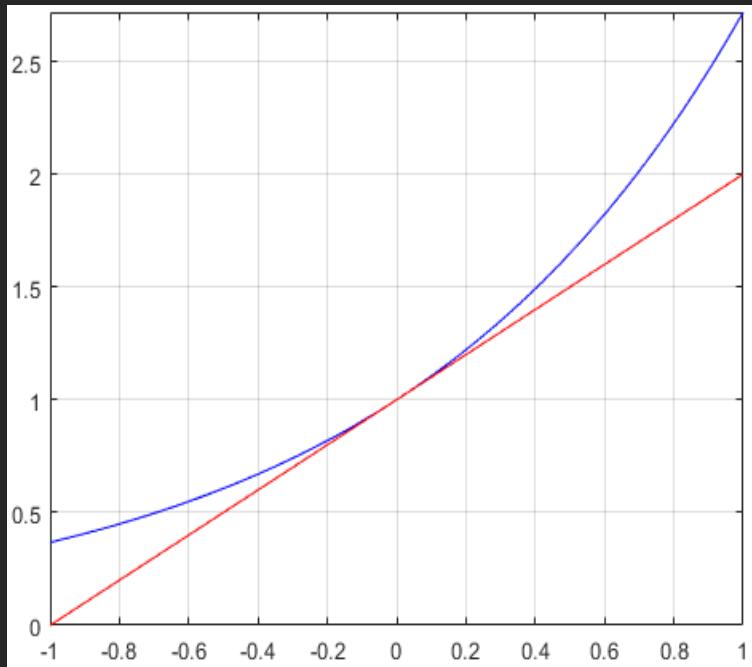
- ▶ The goal is find $y(t_1)$, where $t_1 > t_0$, for a given IVP $y' = F(y, t)$, $y(t_0) = y_0$
 - ▶ Start at the initial time t_0 and the initial value y_0 .
 - ▶ Compute a sequence of values $y(t_0 + \Delta t), y(t_0 + 2\Delta t), \dots, y(t_0 + n\Delta t)$, where $t_1 = t_0 + n\Delta t$.
- ▶ There are two kinds of methods:
 - ▶ explicit methods:
 - ▶ Compute $y(t_0 + \Delta t)$ by a function $\mathcal{F}(F, y_0, t_0, \Delta t)$ of current state of the system.
 - ▶ implicit methods:
 - ▶ Compute $y(t_0 + \Delta t)$ by a solution of an equation $\mathcal{F}(F, y_0, t_0, \Delta t, y(t_0 + \Delta t)) = 0$ over the current and future state of the system.

Taylor theorem

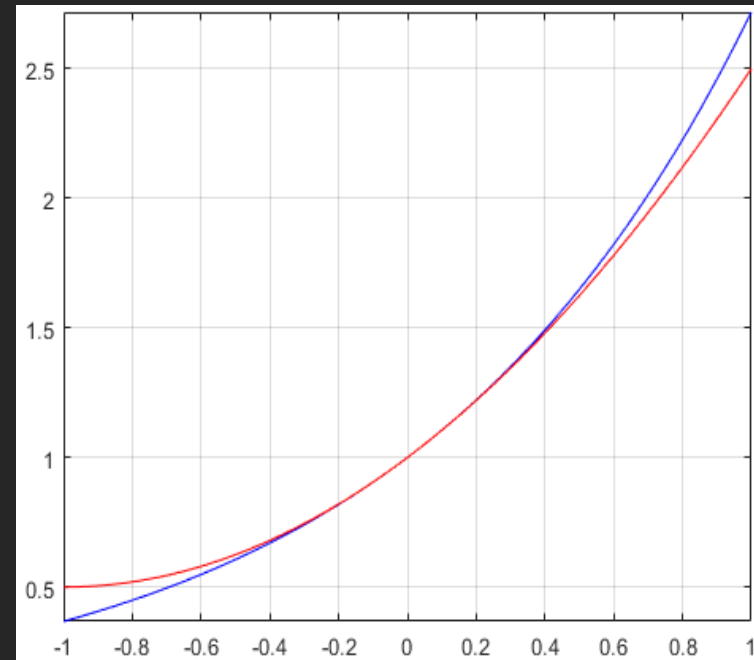
- ▶
- ▶ For a k -times differentiable function $y: \mathbb{R} \rightarrow \mathbb{R}$ at a point $t_0 \in D(y)$ there exists a polynomial $P_k: \mathbb{R} \rightarrow \mathbb{R}$ and a function $R: \mathbb{R} \rightarrow \mathbb{R}$ s.t.
 - ▶ $y(t_0 + \Delta t) = P_k(t_0 + \Delta t) + \Delta t^k R(t_0 + \Delta t)$
 - ▶ $P_k(t_0 + \Delta t) = y(t_0) + \Delta t y'(t_0) + \frac{\Delta t^2}{2!} y''(t_0) + \dots + \frac{\Delta t^k}{k!} y^{(k)}(t_0)$
 - ▶ $\lim_{\Delta t \rightarrow 0} R(t_0 + \Delta t) = 0$

Numerical solution

► Examples (Taylor approximation):



$$y(t) = e^t \quad P_1(t) = 1 + t$$



$$y(t) = e^t \quad P_2(t) = 1 + t + \frac{t^2}{2}$$

"O" error notation

- ▶ What is the error from the approximation using P_k :

$$y(t_0 + \Delta t) \approx P_k(t_0 + \Delta t)$$

- ▶ It's a distance from the exact value $P_{k+1}(t_0 + \Delta t) + \Delta t^{k+1}R(t_0 + \Delta t)$:

$$\text{error} = P_{k+1}(t_0 + \Delta t) + \Delta t^{k+1}R(t_0 + \Delta t) - P_k(t_0 + \Delta t)$$

$$= \frac{\Delta t^{k+1}}{(k+1)!} y^{(k+1)}(t_0) + \Delta t^{k+1}R(t_0 + \Delta t)$$

- ▶ For small Δt the error is proportional to the term Δt^{k+1} . Therefore,

$$y(t_0 + \Delta t) = P_k(t_0 + \Delta t) + O(\Delta t^{k+1})$$

Forward Euler's method

- ▶ We get forward Euler's method, when we approximate y by P_1 at t_0 :

$$\begin{aligned}y(t_0 + \Delta t) &\approx y(t_0) + \Delta t y'(t_0) \\ &= y(t_0) + \Delta t F(y(t_0), t_0)\end{aligned}$$

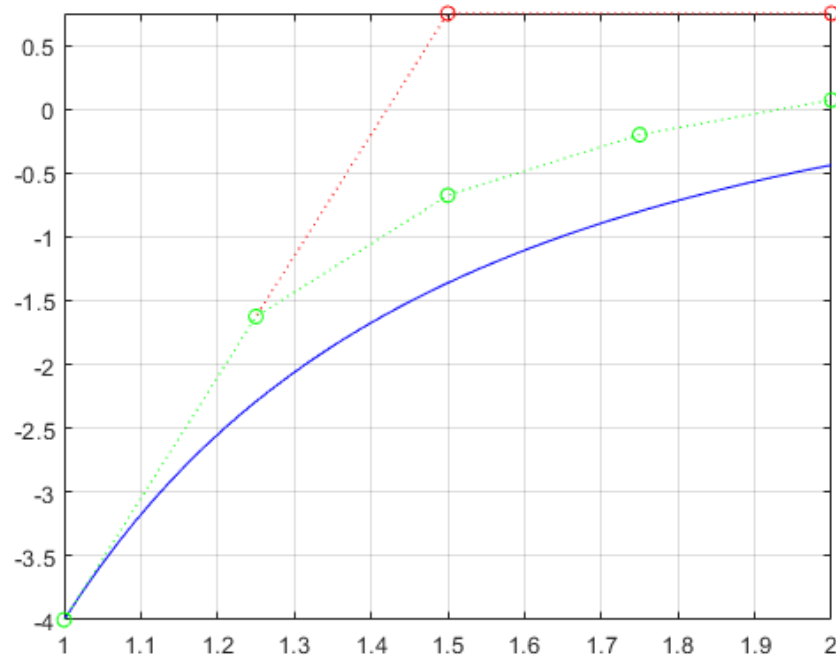
- ▶ We see that forward Euler's method is an explicit method.

Forward Euler's method

- ▶ Example: Let NPV be $y = \frac{3-t^2}{2}$, $y(1) = -4$. Compute $y(2)$ by forward Euler's method. (Note: Exact solution is $y(t) = \frac{3}{2} - \frac{t^2}{2}$)
- ▶ Solution:
 - ▶ Let's choose a time step $\Delta t = \frac{1}{2} \Rightarrow$ We must apply the method 2 times.
 - ▶ $y(1) = -4$ ← from the initial condition.
 - ▶ $y\left(\frac{3}{2}\right) = y\left(1 + \frac{1}{2}\right) = y(1) + \frac{1}{2}F(y(1), 1) = -4 + \frac{1}{2} \cdot \frac{3-4(-4)}{2} = \frac{3}{4}$ ← 1st iteration
 - ▶ $y(2) = \frac{3}{4} + \frac{1}{2} \cdot \frac{3-4\left(\frac{3}{4}\right)}{2} = \frac{3}{4}$ ← 2nd iteration
- ▶ We see the method is simple and fast.

Forward Euler's method

- ▶ Low accuracy issue
- ▶ $O(\Delta t)$ error in each iteration.
- ▶ Example



Exact solution

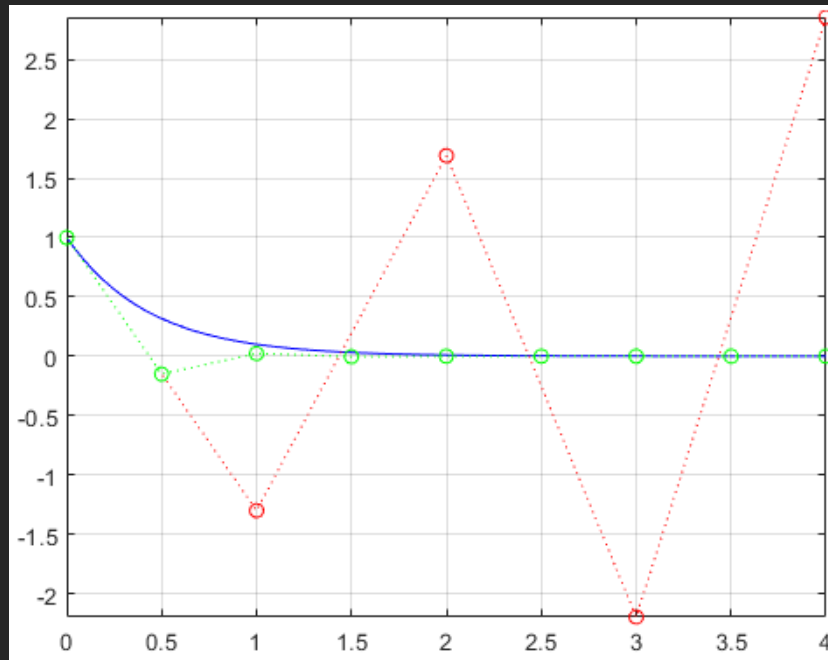
Numerical solution

Exact solution

$$\Delta t = \frac{1}{2}, \Delta t = \frac{1}{4}$$
$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

Forward Euler's method

- ▶ Instability issue:
 - ▶ The iteration process may diverge.
- ▶ **Example:**



$$\Delta t = 1, \Delta t = \frac{1}{2}$$
$$y(t) = e^{-2.3t}$$

Forward Euler's method

- ▶ What can we do with the issues?
 - ▶ Use smaller time step Δt to reduce the error and/or avoid the instability.
 - ▶ But we then need more iterations \rightarrow slower simulation.
 - ▶ Choose more accurate/stable solver.

- ▶ Suggestion for seminar: Implement method "ODE_Euler_forward".

```
void ODE_Euler_forward(  
    std::vector<float> &x0, // x, y of particle(s)  
    std::vector<const float> &Fyt, // F(x, y) of particle(s), i.e. a, F/m  
    float &t, // current time (to be updated)  
    float const dt, // time step  
    std::vector<float> &y) // integrated x, y of particle(s)
```

TODO

Forward Euler's method

- ▶ Example: Let's consider a particle $\mathcal{P}(t) = (x, v, F, m)$, where $m = 0.1\text{kg}$, in a homogenous gravity field with $g = (0, 0, -10)^T \text{m}\cdot\text{s}^{-2}$. At time $t = 1\text{s}$ we have $x = (1, -1.5)^T \text{m}$, $v = (1, 0, 0)^T \text{m}\cdot\text{s}^{-1}$. Using forward Euler's method with $\Delta t = 0.5\text{s}$ compute $\mathcal{P}(2)$.

- ▶ Solution: Particle moves by Newton's equations of motion:

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = \frac{F}{m}$$

$$\text{Therefore: } x(1.5) = \begin{pmatrix} 1 + 0.5 \cdot 1 \\ -1 + 0.5 \cdot 0 \\ 5 + 0.5 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1 \\ 5 \end{pmatrix}, \quad v(1.5) = \begin{pmatrix} 1 + 0.5 \frac{0.1 \cdot 0}{0.1} \\ 0 + 0.5 \frac{0.1 \cdot 0}{0.1} \\ 0 + 0.5 \frac{0.1 \cdot (-10)}{0.1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}.$$

$$x(2) = (2, -1.0)^T, \quad v(2) = (1.0, -10)^T.$$

Backward Euler's method

Backward Euler's method

- ▶ From the fundamental theorem of the calculus:

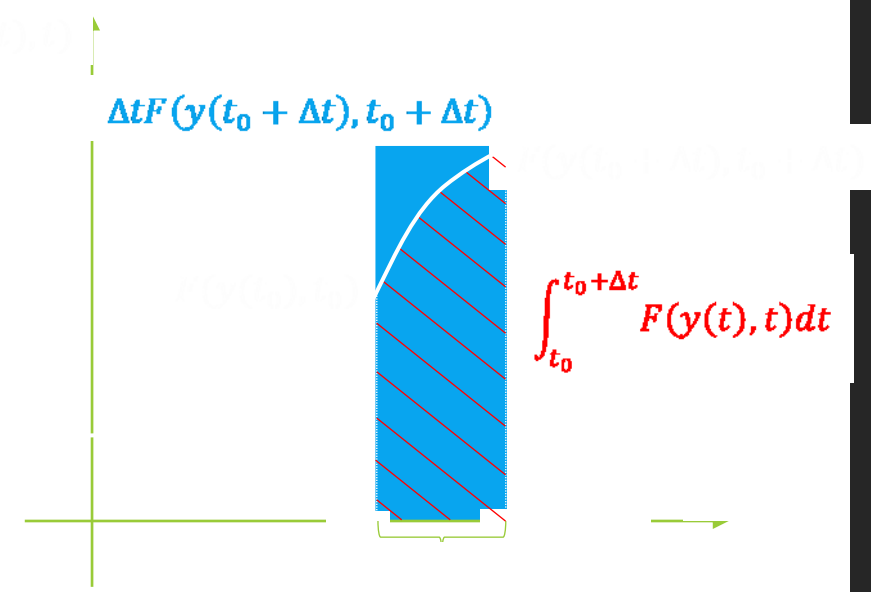
$$\int_{t_0}^{t_0+\Delta t} y'(t) dt = y(t_0 + \Delta t) - y(t_0).$$

- ▶ Therefore,

$$y(t_0 + \Delta t) = y(t_0) + \int_{t_0}^{t_0+\Delta t} F(y(t), t) dt.$$

- ▶ We can approximate the integral by "right-hand" rectangle:

$$\int_{t_0}^{t_0+\Delta t} F(y(t), t) dt \approx \Delta t F(y(t_0 + \Delta t), t_0 + \Delta t).$$



Backward Euler's method

- ▶ Backward Euler's method leads to this equation:

$$y(t_0 + \Delta t) = y(t_0) + \Delta t F(y(t_0 + \Delta t), t_0 + \Delta t)$$

- ▶ Backward Euler's method is an implicit method.

- ▶ We must solve this equation to obtain the unknown $y(t_0 + \Delta t)$:

$$y(t_0 + \Delta t) - y(t_0) - \Delta t F(y(t_0 + \Delta t), t_0 + \Delta t) = 0$$

- ▶ Use any available method for solving the equation, e.g., Newton's method ($y^{[k+1]} = y^{[k]} - f'(y^{[k]})/f''(y^{[k]})$).

- ▶ Note: If we have system of ODEs, then we get system of equations:

Backward Euler's method

▶ Example: Let IVP be $y' = \frac{3-y^2}{2t}$, $y(1) = -4$. Compute $y(2)$ by backward Euler's method. (Note: Exact solution is $y(t) = \frac{3}{t} - \frac{19}{2t^2}$)

▶ Solution:

▶ Let's choose a time step $\Delta t = \frac{1}{2} \Rightarrow$ We must apply the method 2 times.

▶ $y(1) = -4 \leftarrow$ from the initial condition.

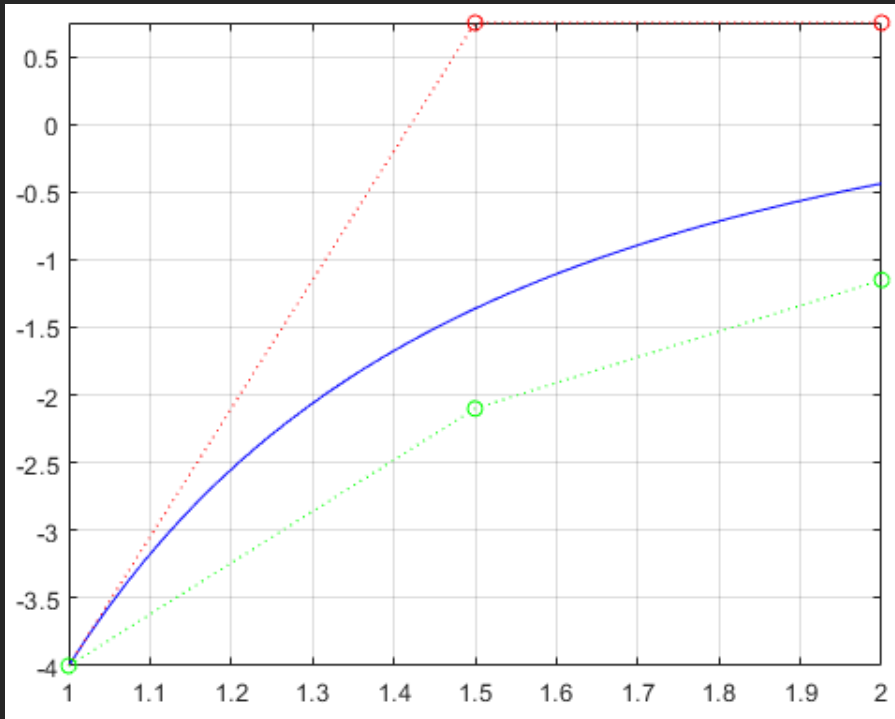
▶
$$y\left(\frac{3}{2}\right) = y\left(1 + \frac{1}{2}\right) = y(1) + \frac{1}{2} F\left(y\left(\frac{3}{2}\right), \frac{3}{2}\right) = -4 + \frac{1}{2} \frac{3-y\left(\frac{3}{2}\right)^2}{2\left(\frac{3}{2}\right)^2} = -\frac{7}{2} - \frac{2}{3} y\left(\frac{3}{2}\right)$$

▶ We solve: $y\left(\frac{3}{2}\right) = -\frac{7}{2} - \frac{2}{3} y\left(\frac{3}{2}\right) \Rightarrow y\left(\frac{3}{2}\right) = -\frac{7}{2\left(1+\frac{2}{3}\right)} = -\frac{21}{10} \leftarrow 1^{\text{st}} \text{ iteration}$

▶
$$y(2) = \frac{21}{10} + \frac{1}{2} \frac{3-y(2)^2}{2 \cdot 2} = \frac{21}{10} + \frac{3}{8} - \frac{1}{8} y(2)^2 \Rightarrow y(2) = -\frac{23}{20} \leftarrow 2^{\text{nd}} \text{ iteration}$$

Backward Euler's method

- We can plot our result and compare it with forward Euler's method:



$$\Delta t = \frac{1}{2}$$
$$\Delta t = \frac{1}{2}$$
$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

Backward Euler's method

▶ Example 2: Let IVP be $y' = -2.3y, y(0) = 1$. Compute $y(4)$ by backward Euler's method.

▶ Solution:

▶ Let's choose a time step $\Delta t = 1 \Rightarrow$ We must apply the method 4 times.

▶ $y(0) = 1 \leftarrow$ from the initial condition.

▶ $y(1) = 1 + 1 \cdot -2.3y(1) \Rightarrow y(1) = \frac{1}{1+2.3} = \frac{10}{33} \leftarrow 1^{\text{st}} \text{ iteration}$

▶ $y(2) = \frac{10}{33} + 1 \cdot -2.3y(2) \Rightarrow y(2) = \frac{10}{33} \cdot \frac{10}{33} = \left(\frac{10}{33}\right)^2 \leftarrow 2^{\text{nd}} \text{ iteration}$

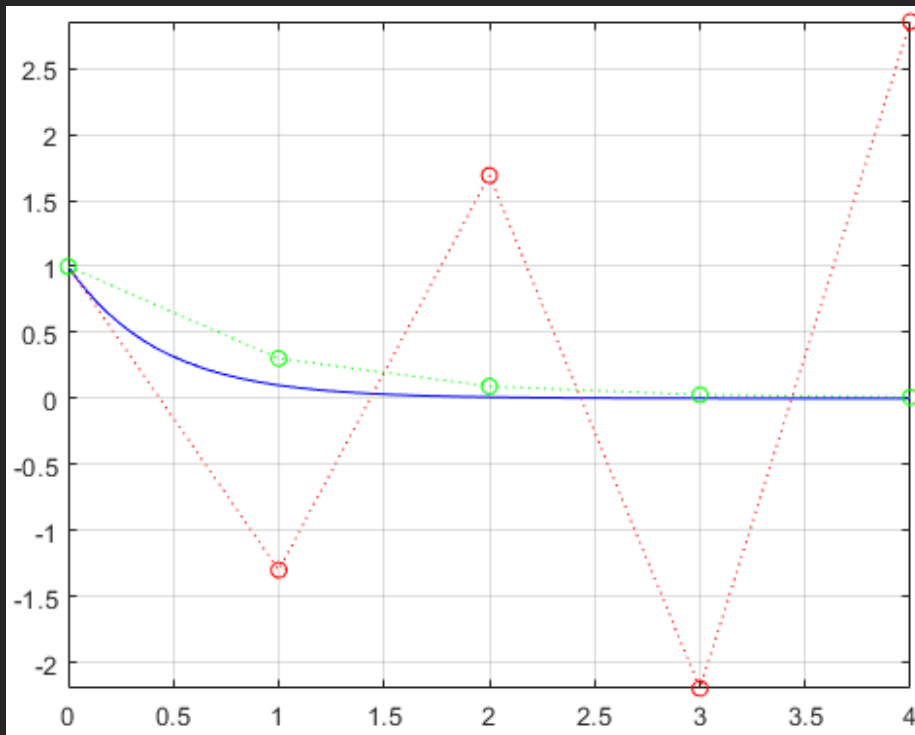
▶ $y(3) = \left(\frac{10}{33}\right)^2 - 2.3y(3) \Rightarrow y(3) = \left(\frac{10}{33}\right)^2 \cdot \frac{10}{33} = \left(\frac{10}{33}\right)^3 \leftarrow 3^{\text{rd}} \text{ iteration}$

▶ $y(4) = \left(\frac{10}{33}\right)^3 - 2.3y(4) \Rightarrow y(4) = \left(\frac{10}{33}\right)^4 \leftarrow 4^{\text{th}} \text{ iteration}$

▶ Note: Observe the geometric progression $y_{k+1} = qy_k, q = \frac{10}{33} \Rightarrow y_k = q^k y_0$.

Backward Euler's method

- We can plot our result and compare it with forward Euler's method:



$$\Delta t = 1$$
$$\Delta t = 1$$
$$y(t) = e^{-2.3t}$$

Backward Euler's method

- ▶ Properties of backward Euler's method
 - ▶ Hard to implement.
 - ▶ Requires solving an equation or a system of equations.
 - ▶ $O(\Delta t^2)$ error in each iteration.
 - ▶ Stable for large time step Δt .
- ▶ Choice between forward/backward Euler's method depends on a problem. "Rule of thumb":
 - ▶ Prefer forward method for "stable" problems.
 - ▶ Prefer backward method for "stiff" problems.

Midpoint method

Midpoint method

- ▶ Let's try to approximate y by P_2 at t_0 :

$$\begin{aligned}y(t_0 + \Delta t) &= y(t_0) + \Delta t y'(t_0) + \frac{\Delta t^2}{2} y''(t_0) + O(\Delta t^3) \\ &= y(t_0) + \Delta t F(y(t_0), t_0) + \underbrace{\frac{\Delta t^2}{2} \dot{F}(y(t_0), t_0)}_{\text{how to compute } \dot{F}} + O(\Delta t^3)\end{aligned}$$

- ▶ How to compute \dot{F} ?

- ▶ Using the chain rule, we get: $\dot{F} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} y' = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} F$

- ▶ Not much better, because we still do not know $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial y}$

- ▶ So, let's try to approximate \dot{F} using P_1 ...

- ▶ Note: We must use a 2-variables version of Taylor's theorem.

Midpoint method

► $F(y(t_0) + \Delta y, t_0 + \Delta t) = F(y(t_0), t_0) + \Delta y \frac{\partial F}{\partial y}(y(t_0), t_0) + \Delta t \frac{\partial F}{\partial t}(y(t_0), t_0) + O(\Delta y^2 + \Delta t^2)$

► Let's substitute: $\Delta y = \frac{\Delta t}{2} F(y(t_0), t_0)$, $\Delta t = \frac{\Delta t}{2}$

$$F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right) = F(y(t_0), t_0) + \underbrace{\frac{\Delta t}{2} F(y(t_0), t_0) \frac{\partial F}{\partial y}(y(t_0), t_0) + \frac{\Delta t}{2} \frac{\partial F}{\partial t}(y(t_0), t_0)}_{+O(\Delta t^2)}$$

$$F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right) = F(y(t_0), t_0) + \frac{\Delta t}{2} F(y(t_0), t_0) + O(\Delta t^2)$$

$$\frac{\Delta t}{2} F(y(t_0), t_0) + O(\Delta t^2) = F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right) - F(y(t_0), t_0)$$

$$\underbrace{\frac{\Delta t^2}{2} F(y(t_0), t_0) + O(\Delta t^3)}_{(*)} = \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right) - \Delta t F(y(t_0), t_0)$$

(*)

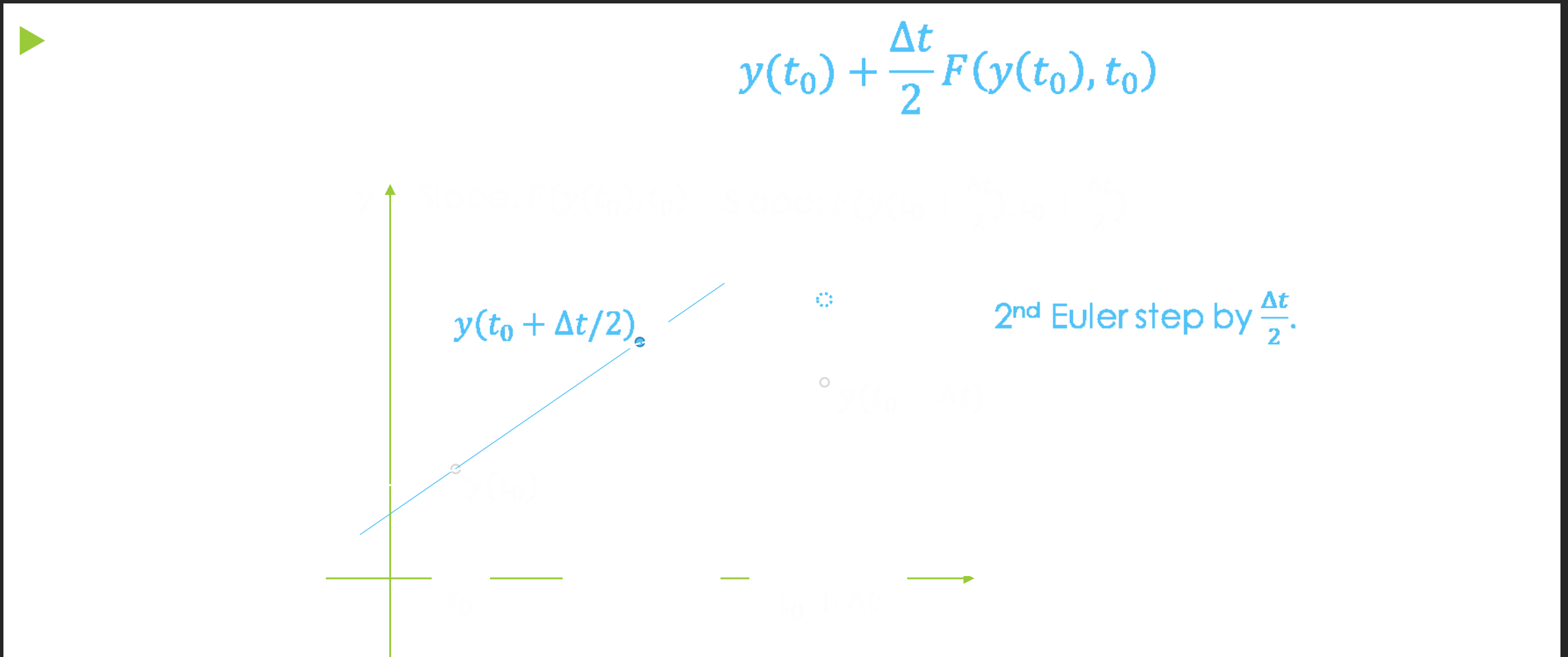
Midpoint method

▶
$$y(t_0 + \Delta t) = y(t_0) + \Delta t F(y(t_0), t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right) - \Delta t F(y(t_0), t_0)$$

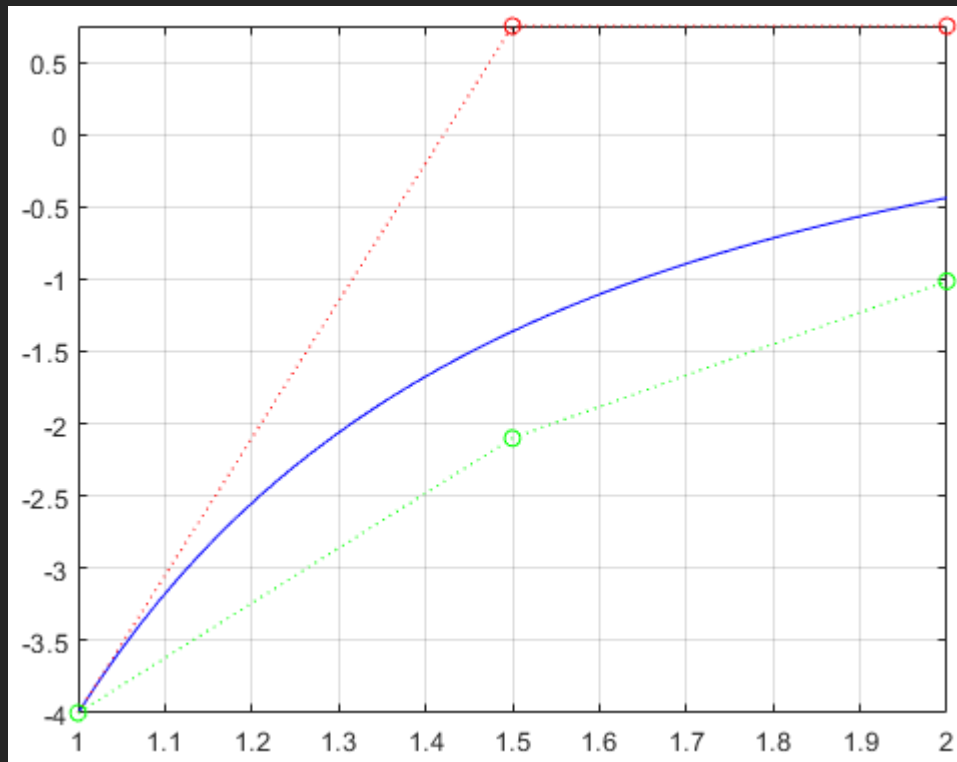
$$y(t_0 + \Delta t) = y(t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right)$$

- ▶ This is an explicit method.
- ▶ This method is more accurate than Euler's method:
 - ▶ Euler: $O(\Delta t^2)$
 - ▶ Midpoint: $O(\Delta t^3)$

Midpoint method

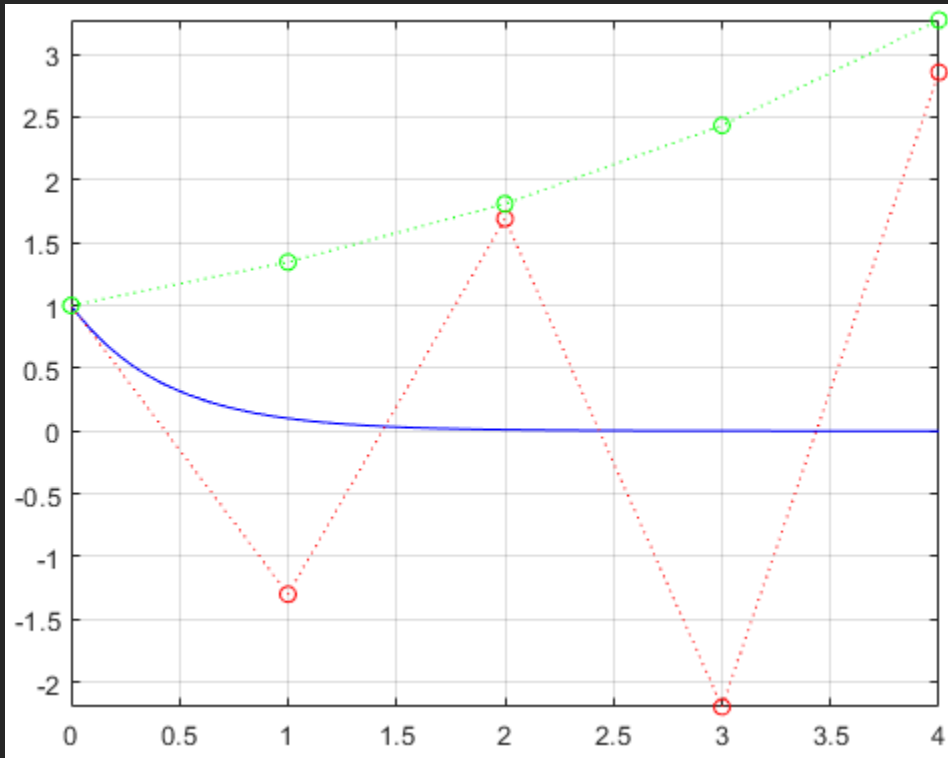


Midpoint method



$$\Delta t = \frac{1}{2}$$
$$\Delta t = \frac{1}{2}$$
$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

Midpoint method



$$\Delta t = 1$$
$$\Delta t = 1$$
$$y(t) = e^{-2.3t}$$

Midpoint method

▶ There is also implicit version of the midpoint method.

▶ From the fundamental theorem of the calculus

$$y(t_0 + \Delta t) = y(t_0) + \int_{t_0}^{t_0 + \Delta t} F(y(t), t) dt.$$

▶ We can approximate the integral by "midpoint" rectangles

$$\int_{t_0}^{t_0 + \Delta t} F(y(t), t) dt \approx \Delta t F\left(\frac{y(t_0) + y(t_0 + \Delta t)}{2}, \frac{t_0 + (t_0 + \Delta t)}{2}\right)$$

▶ Therefore, we get

$$y(t_0 + \Delta t) = y(t_0) + \Delta t F\left(\frac{y(t_0) + y(t_0 + \Delta t)}{2}, t_0 + \frac{\Delta t}{2}\right)$$

Runge-Kutta methods

Runge-Kutta methods

- ▶ In general, we can approximate the integral as follows:

$$\int_{t_0}^{t_0+\Delta t} F(y(t), t) dt \approx \Delta t \sum_{i=1}^n b_i F(y(t_0 + c_i \Delta t), t_0 + c_i \Delta t)$$

- ▶ The problem is that values $y(t_0 + c_i \Delta t)$ are unknown!
- ▶ Runge-Kutta methods solve the issue by this substitution:

$$k_1 = F(y(t_0), t_0)$$

$$k_i = F\left(y(t_0) + \Delta t \sum_{j=1}^{i-1} a_{ij} k_j, t_0 + c_i \Delta t\right), \quad \text{s.t. } \sum_{j=1}^{i-1} a_{ij} = c_i$$

$$\int_{t_0}^{t_0+\Delta t} F(y(t), t) dt \approx \Delta t \sum_{i=1}^n b_i k_i$$

Runge-Kutta methods

- ▶ Therefore, Runge-Kutta of order n is defined as:

$$y(t_0 + \Delta t) = y(t_0) + \Delta t \sum_{i=1}^n b_i k_i,$$

where terms k_i were defined on the previous slide.

- ▶ However, we must compute the numbers a_{ij} , b_i , c_i so that resulting expression yields an approximation by Taylor's polynomial P_n .

Runge-Kutta methods

- ▶ Example: Runge-Kutta method of order 1 (i.e. $\kappa = 1$):

$$k_1 = F(y(t_0), t_0)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t b_1 k_1 = y(t_0) + \Delta t b_1 F(y(t_0), t_0)$$

What value we should choose for b_1 ? We compare $y(t_0 + \Delta t)$ with P_1 .

$$P_1(t_0 + \Delta t) = y(t_0) + \Delta t y'(t_0) = y(t_0) + \Delta t F(y(t_0), t_0)$$

Therefore, b_1 must be 1.

- ▶ Observation: Euler's method is Runge-Kutta method of order 1.

Runge-Kutta methods

► Example: Runge-Kutta method of order 2 (i.e. $\pi = 2$):

$$k_1 = F(y(t_0), t_0)$$

$$k_2 = F(y(t_0) + \Delta t a_{2,1} k_1, t_0 + c_2 \Delta t)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t b_1 k_1 + \Delta t b_2 k_2$$

We compute $a_{2,1}, b_1, b_2$ by comparison of $y(t_0 + \Delta t)$ with P_2 .

$$P_2(t_0 + \Delta t) = y(t_0) + \Delta t y'(t_0) + \frac{\Delta t^2}{2!} y''(t_0)$$

$$= y(t_0) + \Delta t F(y(t_0), t_0) + \frac{\Delta t^2}{2} F'(y(t_0), t_0)$$

$$= y(t_0) + \Delta t F(y(t_0), t_0) + \frac{\Delta t^2}{2} \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} F \right) (y(t_0), t_0)$$

Runge-Kutta methods

- For the comparison let's approximate k_2 by P_2 :

$$\begin{aligned}k_2 &= F(y(t_0) + \Delta t a_{2,1} k_1, t_0 + c_2 \Delta t) \\ &= F(y(t_0), t_0) + \Delta t \left(c_2 \frac{\partial F}{\partial t} + a_{2,1} k_1 \frac{\partial F}{\partial y} \right) (y(t_0), t_0) \\ &= F(y(t_0), t_0) + \Delta t \left(c_2 \frac{\partial F}{\partial t} + a_{2,1} F \frac{\partial F}{\partial y} \right) (y(t_0), t_0)\end{aligned}$$

When we substitute the approximated k_2 we get:

$$\begin{aligned}y(t_0 + \Delta t) &= y(t_0) + \Delta t b_1 F(y(t_0), t_0) \\ &\quad + \Delta t b_2 \left(F(y(t_0), t_0) + \Delta t \left(c_2 \frac{\partial F}{\partial t} + a_{2,1} F \frac{\partial F}{\partial y} \right) (y(t_0), t_0) \right) \\ &= y(t_0) + \Delta t (b_1 + b_2) F(y(t_0), t_0) \\ &\quad + \Delta t^2 b_2 \left(c_2 \frac{\partial F}{\partial t} + a_{2,1} F \frac{\partial F}{\partial y} \right) (y(t_0), t_0)\end{aligned}$$

Runge-Kutta methods

► So, we must solve this system of equations:

$$b_1 + b_2 = 1, \quad b_2 c_2 = \frac{1}{2}, \quad b_2 a_{2,1} = \frac{1}{2}.$$

One possible solution is: $b_1 = 0, b_2 = 1, c_2 = \frac{1}{2}, a_{2,1} = \frac{1}{2}$.

(Note: Another solution is: $b_1 = \frac{1}{2}, b_2 = \frac{1}{2}, c_1 = 1, a_{2,1} = 1$)

We get the result:

$$k_1 = F(y(t_0), t_0)$$

$$k_2 = F\left(y(t_0) + \frac{\Delta t}{2} k_1, t_0 + \frac{\Delta t}{2}\right)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t k_2 = y(t_0) + \Delta t F\left(y(t_0) + \frac{\Delta t}{2} F(y(t_0), t_0), t_0 + \frac{\Delta t}{2}\right).$$

► Observation: Midpoint method is Runge-Kutta method of order 2.

Runge-Kutta methods

► Example: Runge-Kutta method of order 4:

$$k_1 = F(y(t_0), t_0)$$

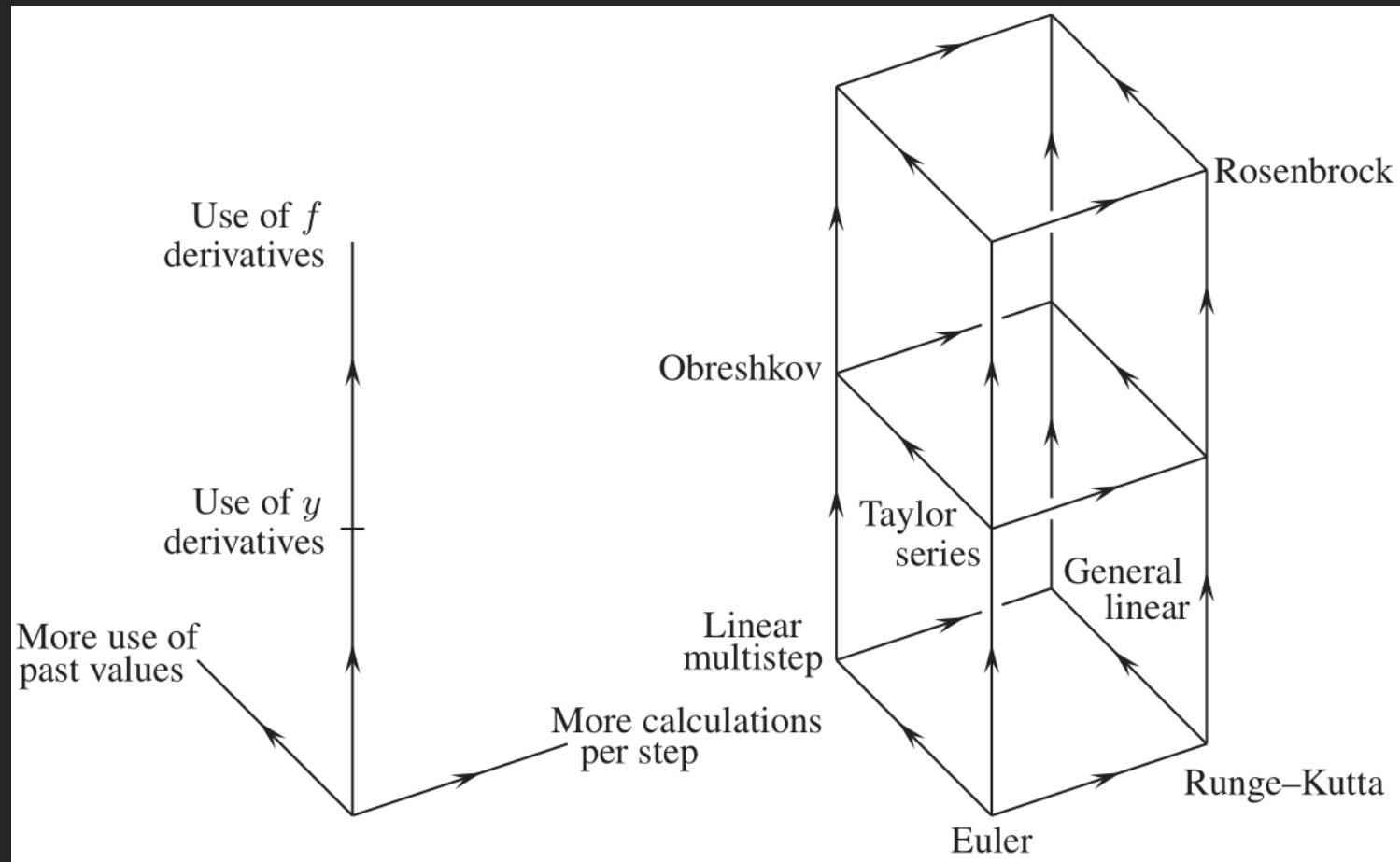
$$k_2 = F\left(y(t_0) + \frac{k_1}{2}, t_0 + \frac{\Delta t}{2}\right)$$

$$k_3 = F\left(y(t_0) + \frac{k_2}{2}, t_0 + \frac{\Delta t}{2}\right)$$

$$k_4 = F(y(t_0) + k_3, t_0 + \Delta t)$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t \left(\frac{1}{6}k_1 + \frac{4}{6}k_2 + \frac{1}{6}k_3 + \frac{1}{6}k_4\right)$$

Schema of numerical methods



Picture source: [2]

References

- [1] A. Witkin, D. Baraff; *Differential Equation Basics; Physically Based Modeling: Principles and Practice*, 1997
- [2] J.C. Butcher; *Numerical methods for ordinary differential equations; 3rd edition*, Wiley, 2016.
- [3] <https://tutorial.math.lamar.edu/Classes/DE/DE.aspx>