Fluid simulation

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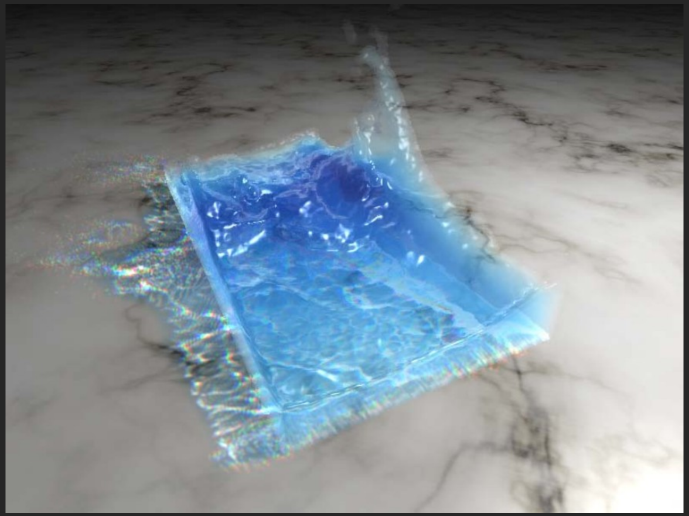
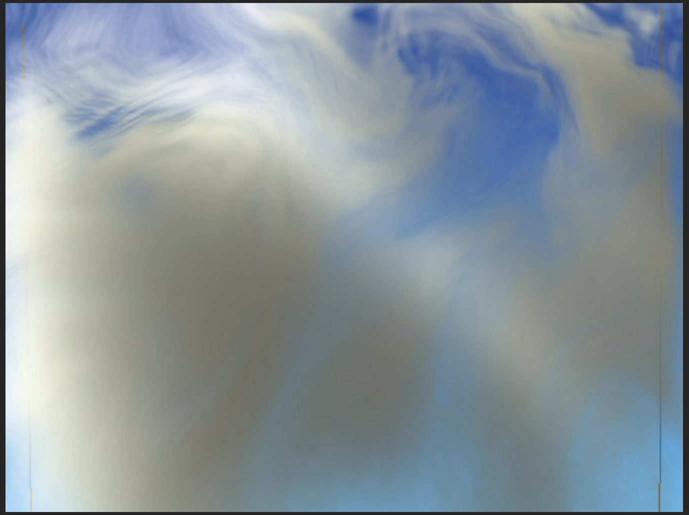
PA199

Motivation

Picture source: [2]





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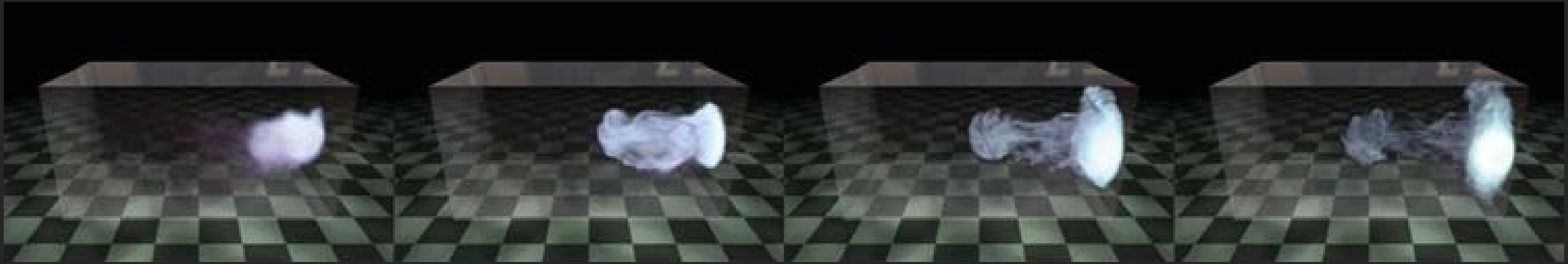


Other pictures source: [5]

Outline

- **Euler approach**:
  - Fluid is modelled by a vector field, representing the velocity of the fluid.
- **Lagrange approach**:
  - Fluid is modelled by set of particles.
- **Smoothed Particle Hydrodynamics**:
  - Fluid is modelled by set of particles moved via a velocity vector field.
- **Hight-field surface approximation**:
  - Suitable for simulation of only fluid’s surface, e.g., lake or ocean surface.

Euler approach

Fluid Model

- Assumptions:
  - Incompressible fluid:
    - Volume of any subregion of the fluid is constant over time.
    - Represented by an incompressible constraint.
  - Homogeneous fluid:
    - The density of fluid is the same and constant in every region of the fluid and over time.
- Navier-Stokes equations model a fluid:
  - Fluid velocity (motion) represented by a vector field $u(x, t)$.
  - Fluid pressure represented by a scalar field $p(x, t)$.
  - Partial differential equations define changes in the vector field u over time.

Navier-Stokes Equations

- The momentum equations (for each coordinate one):

$$\frac{\partial u}{\partial t} = \underbrace{-(u \cdot \nabla)u}_{} \underbrace{- \frac{1}{\rho}\nabla p}_{} \underbrace{+ \nu \nabla^2 u}_{} \underbrace{+ g}_{}$$

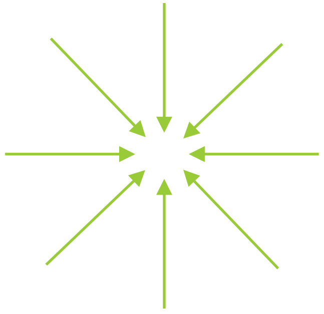
- The incompressibility constraint:

$$\nabla \cdot u = 0$$

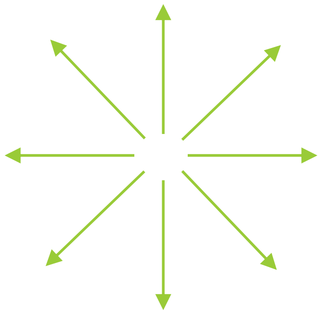
- Where (let $x = (x, y, z)^T$ be a position in space and t be simulation time):
  - $u(x, t) = (u(x, t), v(x, t), w(x, t))^T$ is the velocity vector field of the fluid **(computed)**
  - $p(x, t)$ is a pressure scalar field of the fluid; used to preserve incompressibility **(computed)**
  - ρ is the density of the fluid, e.g., water $\rho = 10^3 \frac{kg}{m^3}$.
  - ν is the viscosity (resistance to deformation) of the fluid, e.g., honey – high viscosity, water – low viscosity.
  - $g(x, t)$ is the acceleration vector field of forces acting on the fluid, e.g., gravity $g(x, t) = (0, 0, -10)^T \frac{m}{s^2}$.
  - $\cdot$ is the dot product.

Gradient, Divergence and Laplacian

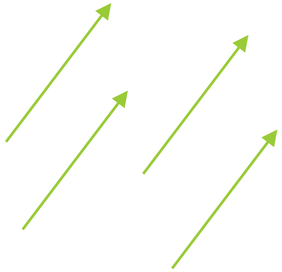
- Operator of spatial partial derivatives: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$
  - Identifies a direction of a maximum increase of a function at a given time
  - Example: $\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)^T$
- Divergence operator: $\nabla \cdot \boldsymbol{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T \cdot (u, v, w)^T = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$
  - Can only be applied to a vector field



$\nabla \cdot \boldsymbol{u} < 0$



$\nabla \cdot \boldsymbol{u} > 0$



$\nabla \cdot \boldsymbol{u} = 0$

Gradient, Divergence and Laplacian

- Directional derivative: $\mathbf{u} \cdot \nabla = (u, v, w)^T \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$
  - Therefore, $(\mathbf{u} \cdot \nabla)\mathbf{u} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u, v, w)^T = \begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix}$
- Laplacian operator: $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
  - Example: $\nabla^2 \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(u, v, w)^T = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}$

Adding Custom Quantities

- Beside the fluid we also simulate other quantities, e.g., smoke density, temperature.
- Represent any such quantity q as another scalar/vector field.
- Add a related equation, how q changes in time:

$$\frac{\partial q}{\partial t} = -(u \cdot \nabla)q + \nu \nabla^2 q + S$$

- Observe the similarity with the momentum equations:
  - Advection: $-(u \cdot \nabla)q$
  - Diffusion: $\nu \nabla^2 q$
  - We do not have pressure term.
  - S can be used to simulate constant inflow of q into the fluid.

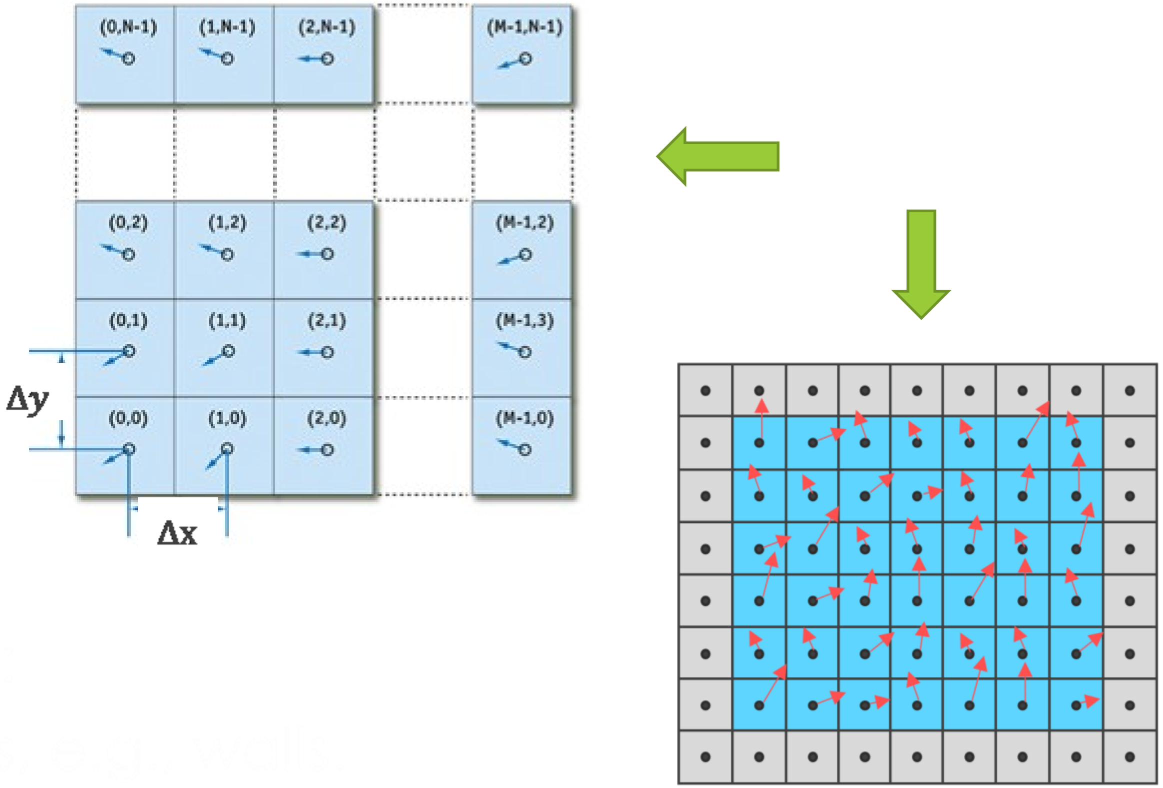
=> Methods for solving the momentum equation can be also applied for q equation.

Boundary Conditions

- The fluid can collide with:
  - Static solid objects, like walls.
  - Freely moveable solid objects, like piece of wood in water.
  - Another fluid, like oil stain on water surface. (not covered in this lecture)
- Our goal is to prevent the fluid to flow into the solid objects.
- Let $n(x,t), u_s(x,t)$ be the normal and velocity of the solid surface.
- The boundary constraint for:
  - Low viscosity fluid: $u(x,t) \cdot n(x,t) = u_s(x,t) \cdot n(x,t)$
  - High viscosity fluid: $u(x,t) = u_s(x,t)$
- We can use boundary condition to model fluid source and/or sink.

Discretize fields

- Discretize the space into a regular grid. For each cell i, j, k we store:
  - Fluid velocity: $u_{i,j,k}$
  - Pressure: $p_{i,j,k}$
  - Any other field: $q_{i,j,k}$



- Discretize boundary conditions:
  - Mark cells filled by solid objects, e.g., walls.

Discretize derivatives

- Use finite differences to approximate partial derivatives.
- Examples:

$$\nabla p_{i,j,k} = \left(\frac{p_{i+1,j,k} - p_{i-1,j,k}}{2\Delta x}, \frac{p_{i,j+1,k} - p_{i,j-1,k}}{2\Delta y}, \frac{p_{i,j,k+1} - p_{i,j,k-1}}{2\Delta z} \right)$$

$$\nabla \cdot w_{i,j,k} = \frac{w_{i+1,j,k} - w_{i-1,j,k}}{2\Delta x} + \frac{w_{i,j+1,k} - w_{i,j-1,k}}{2\Delta y} + \frac{w_{i,j,k+1} - w_{i,j,k-1}}{2\Delta z}$$

$$\nabla^2 q_{i,j,k} = \frac{q_{i+1,j,k} - 2q_{i,j,k} + q_{i-1,j,k}}{\Delta x^2} + \frac{q_{i,j+1,k} - 2q_{i,j,k} + q_{i,j-1,k}}{\Delta y^2} + \frac{q_{i,j,k+1} - 2q_{i,j,k} + q_{i,j,k-1}}{\Delta z^2}$$

Solving Equations

- Method of splitting:
  - Solve a complex equation by a sequence numerical integrations:

$$\frac{dq}{dt} = f(q) + g(q) \quad \rightarrow \quad \begin{aligned} \tilde{q} &= q^t + \Delta t f(q^t) \\ q^{t+\Delta t} &= \tilde{q} + \Delta t g(\tilde{q}) \end{aligned}$$

  - The result is equivalent to a single integration:

$$\begin{aligned} q^{t+\Delta t} &= \tilde{q} + \Delta t g(\tilde{q}) \\ &= q^t + \Delta t f(q^t) + \Delta t g(q^t + \Delta t f(q^t)) \\ &= q^t + \Delta t f(q^t) + \Delta t (g(q^t) + O(\Delta t)) \\ &= q^t + \Delta t (f(q^t) + g(q^t)) + O(\Delta t^2) \\ &= q^t + \Delta t \frac{dq}{dt} + O(\Delta t^2) \end{aligned}$$

Solving Equations

- We solve the momentum equation using the splitting method:

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \frac{1}{\rho}\nabla p + \nu \nabla^2 u + g$$

  - Start in the current state:

    $w_0(x) = u(x, t)$

  - Apply external accelerations g:

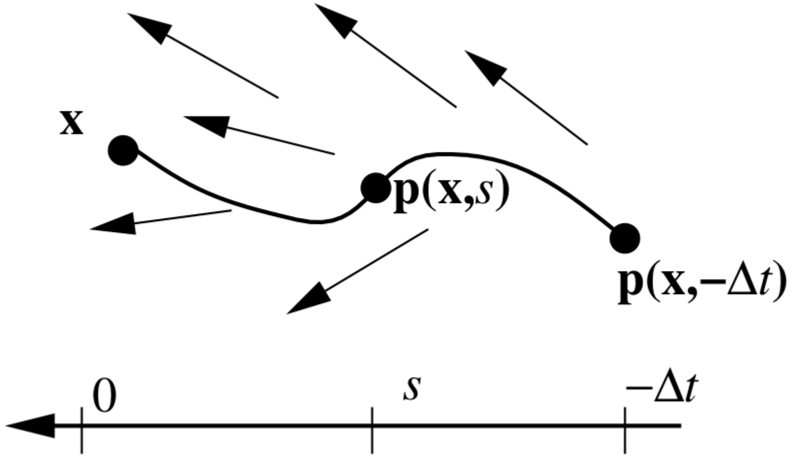
    $w_1(x) = w_0(x) + \Delta t g$

    (forward Euler)

  - Apply fluid advection $-(u \cdot \nabla)u$:

    $w_2(x) = w_1(p(x, -\Delta t))$

    (method of characteristics)



The new velocity at x is the velocity that the particle had a time Δt ago at the location $p(x, -\Delta t)$ (going backward in time along u)

Solving Equations

- We solve the momentum equation using the splitting method:

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \frac{1}{\rho}\nabla p + \nu \nabla^2 u + g$$

  - Apply fluid viscosity $\nu \nabla^2 u$:

$$w_3(x) = w_2(x) + \Delta t \nu \nabla^2 w_3(x)$$

(backward Euler)

- Lastly, we must compute the pressure $-\frac{1}{\rho}\nabla p$ acceleration to remove divergence from w_3, i.e., to satisfy the incompressibility:

$$\nabla \cdot u = 0$$

Solving Equations

- Helmholtz-Hodge Decomposition: Any vector field w can be uniquely decomposed to a vector field u and a scalar field p satisfying:

$$w = u + \nabla p$$

where u is a divergence free, i.e., $\nabla \cdot u = 0$.

- When we apply divergence operator to both sides of the equation:

$$\nabla \cdot w = \nabla^2 p$$

we get a Poisson equation.

- Due to discretization, we get a sparse system of linear equations

=> Use, for example, Jacobi method.

- We use the computed pressure field to get the resulting fluid velocity:

$$u(x, t + \Delta t) = w_3(x) - \nabla p$$

Euler approach

DEMO!

https://paveldogreat.github.io/WebGL-Fluid-Simulation/
http://haxiomic.github.io/projects/webgl-fluid-and-particles/

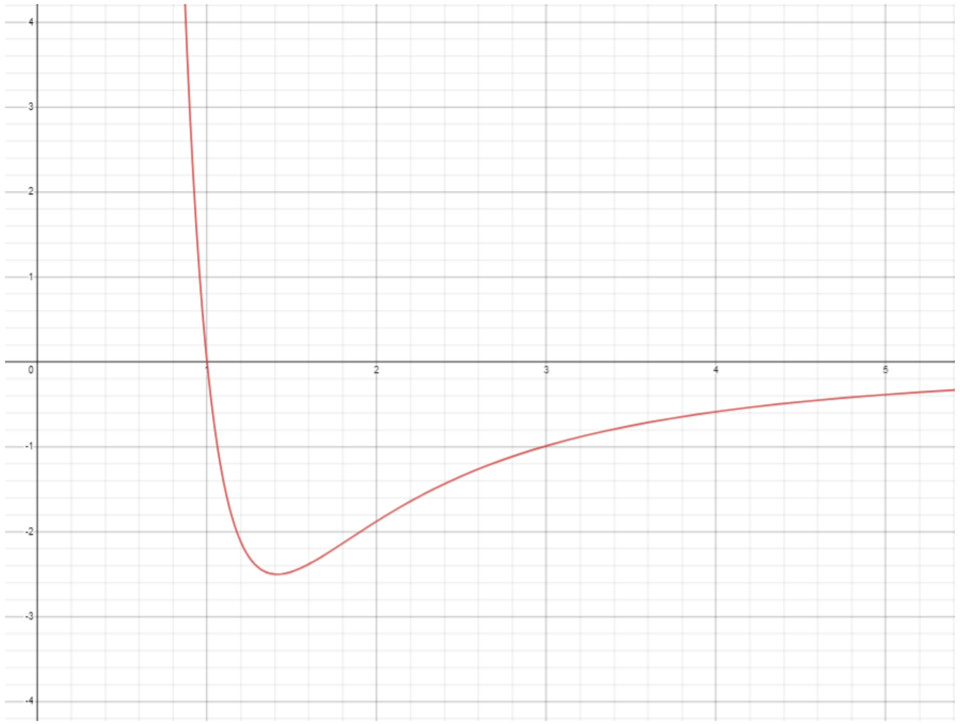
Lagrange approach

Particles Simulation

- The fluid is represented by n particles $\{\mathcal{P}_0, \dots, \mathcal{P}_{n-1}\}$.
- Each particle $\mathcal{P}_i$ is defined by:
  - Mass: m_i
  - Position vector: p_i
  - Velocity vector: u_i
  - Total external force: f_i
- Newton's equations of motion for moving particles:
  - $\frac{dp_i}{dt} = u_i$ (3 equations in 3D space)
  - $\frac{du_i}{dt} = \frac{f_i}{m_i}$ (3 equations in 3D space)

External Forces

- The attribute f_i of a particle $\mathcal{P}_i$ is a sum of all forces acting on the particle
- We usually want Earth's gravity to act on particles:
  - Force of a homogenous field: $m_i g$
  - Typically: $g = (0, 0, -10)^T$
- Interaction between particles $\mathcal{P}_i$ and $\mathcal{P}_j$ via Lennard-Jones force:
  - Let $d_{ij} = |p_i - p_j|$ and $\hat{d}_{ij} = \frac{p_i - p_j}{d_{ij}}$.
  - $F_{ij} = \left(\frac{k_1}{d_{ij}^m} - \frac{k_2}{d_{ij}^n}\right) \hat{d}_{ij}, \quad F_{ji} = -F_{ij}$
  - where typically $k_1 = k_2$, $m = 4$ and $n = 2$.



$|F_{ij}|$ for $k_1 = k_2 = 1.0$, $m = 4$, $n = 2$.

Plotted at https://www.desmos.com/calculator

Lagrange approach

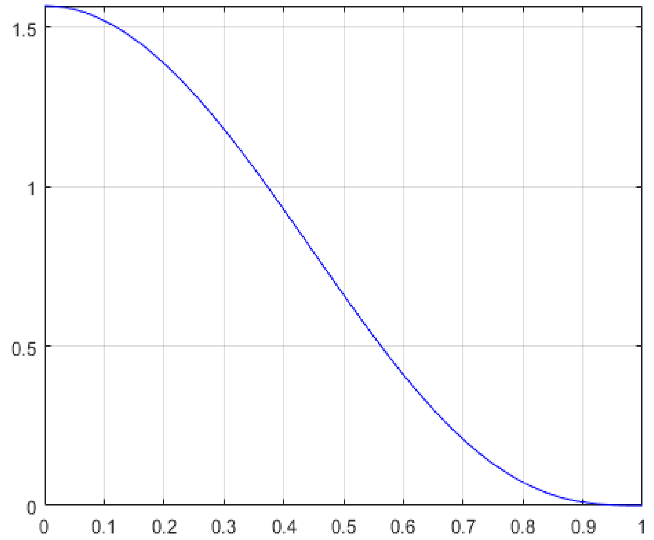
DEMO!

Smoothed Particle Hydrodynamics

Smoothed Particle Hydrodynamics

- Simulate fluid using a set of n particles, i.e., Lagrange approach.
- Compute forces acting on the particles by Euler approach. How?
- Smooth properties of particles into continuous fields.
  - Use a smoothing kernel $W(x)$, e.g., poly6:

$$W(x) = \frac{315}{64\pi d^9}\begin{cases}(d^2 - x^2)^3 & \text{if } 0 \le x \le d\\ 0 & \text{otherwise}\end{cases}$$



  - Let A be a property of particle. Then continuous field $A(x)$ is:

$$A(x) = \sum_{j=0}^{n-1} m_j \frac{A_j}{\rho_j} W(|x_j - x|).$$

    - Example: $\rho(x) = \sum_{j=0}^{n-1} m_j \frac{\rho_j}{\rho_j} W(|x_j - x|) = \sum_{j=0}^{n-1} m_j W(|x_j - x|)$.

Smoothed Particle Hydrodynamics

- With the fields defined we can use momentum and incompressibility equations:

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \frac{1}{\rho}\nabla p + \nu \nabla^2 u + g, \qquad \nabla \cdot u = 0.$$

- We simulate particles => mass is conserved => $\nabla \cdot u = 0$ is not needed.
- Particles automatically move with the fluid => $-(u \cdot \nabla)u$ is not needed.
- So, we only solve: $\frac{\partial u}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u + g$.
- Recall second Newton's equation of motion: $\frac{du_i}{dt} = \frac{f_i}{m_i}$
- Therefore, $\frac{f_i}{m_i} = -\frac{1}{\rho(x_i)}\nabla p(x_i) + \nu \nabla^2 u(x_i) + g$.

Smoothed Particle Hydrodynamics

- The pressure field p can be obtained from density field ρ by law of ideal gas:
  - $p(x) = k(\rho(x) - \rho_0)$, where k is a gas constant and ρ_0 is the environment pressure
- Derivatives of any field $A(x)$:

$$\nabla A(x) = \sum_{j=0}^{n-1} m_j \frac{A_j}{\rho_j} \nabla W(|x_j - x|), \quad \nabla^2 A(x) = \sum_{j=0}^{n-1} m_j \frac{A_j}{\rho_j} \nabla^2 W(|x_j - x|)$$

where $\nabla W(|x_j - x|) = W'(|x_j - x|)\frac{x_j - x}{|x_j - x|}$, $\nabla^2 W(|x_j - x|) = W''(|x_j - x|) + \frac{2W'(|x_j - x|)}{|x_j - x|}$

- Forces between two particles generated by fields ∇p, $\nabla^2 u$ should be symmetric ⇒ we usually modify their computation:

$$\nabla p(x_i) = \sum_{j=0}^{n-1} m_j \frac{p_i + p_j}{2\rho_j} \nabla W(|x_j - x_i|), \quad \nabla^2 u(x_i) = \sum_{j=0}^{n-1} m_j \frac{u_j - u_i}{\rho_j} \nabla^2 W(|x_j - x_i|)$$

Height-field surface approximation

Fluid Surface Model

- We model a fluid surface by a function $h(x, y, t)$.
  - At a point (x, y) in the XY plane and in time t the function defines fluid height $z = h(x, y, t)$.
- Change of h in time is given by:

  $$\frac{\partial^2 h}{\partial t^2} = v^2 \nabla^2 h$$

  where v is the speed of waves in the fluid.
- How to solve the equation?
  - Introduce an auxiliary function $q = \frac{\partial h}{\partial t}$
  - Rewrite the equation into this system:

    $$\frac{\partial q}{\partial t} = v^2 \nabla^2 h, \qquad \frac{\partial h}{\partial t} = q.$$

  - Discretize (next slide).

Discretize Model

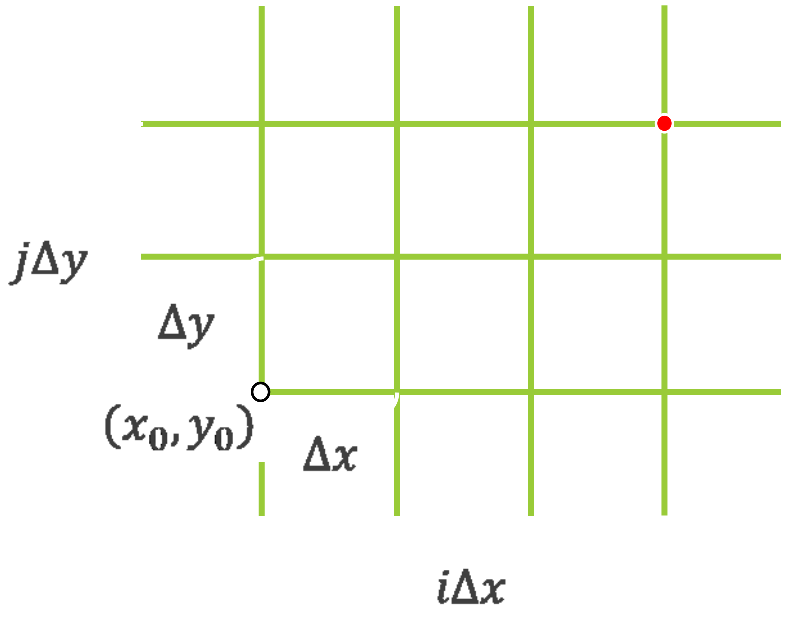
- We discretize the functions h, q by 2D arrays:

  $h(x_0 + i\Delta x, y_0 + j\Delta y, t_0 + k\Delta t) => h_{i,j}^k$

  $q(x_0 + i\Delta x, y_0 + j\Delta y, t_0 + k\Delta t) => q_{i,j}^k$

  where

  - i, j are indices to the arrays.
  - $\Delta x, \Delta y$ are distances between grid cells in X,Y axes.
  - k simulation step number.
  - Δt simulation time step.
  - NOTE: Usually, $x_0 = y_0 = t_0 = 0$.



- We solve the discretized system numerically, e.g., using forward Euler method:

$$q_{i,j}^{k+1} = q_{i,j}^k + \Delta t v^2 \left(\frac{h_{i+1,j}^k - 2h_{i,j}^k + h_{i-1,j}^k}{\Delta x^2} + \frac{h_{i,j+1}^k - 2h_{i,j}^k + h_{i,j-1}^k}{\Delta y^2} \right),$$

$$h_{i,j}^{k+1} = h_{i,j}^k + \Delta t q_{i,j}^{k+1}.$$

Hight-field surface approximation

DEMO!

References

[1] W.J. Laan, S. Green, M. Sainz; Screen Space Fluid Rendering with Curvature Flow; I3D 2009.

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