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# Basics in Language and Probability

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# Quotes



It must be recognized that the notion "probability of a sentence" is an entirely useless one, under any known interpretation of this term.

Noam Chomsky, 1969

Whenever I fire a linguist our system performance improves.

Frederick Jelinek, 1988

# Conflicts?



rationalist vs. empiricist

scientist vs. engineer

insight vs. data analysis

explaining language vs. building applications

# language

# A Naive View of Language



- Language needs to name
  - nouns: objects in the world (**dog**)
  - verbs: actions (**jump**)
  - adjectives and adverbs: properties of objects and actions (**brown, quickly**)
- Relationship between these have to specified
  - word order
  - morphology
  - function words

# A Bag of Words



**quick**

**fox**

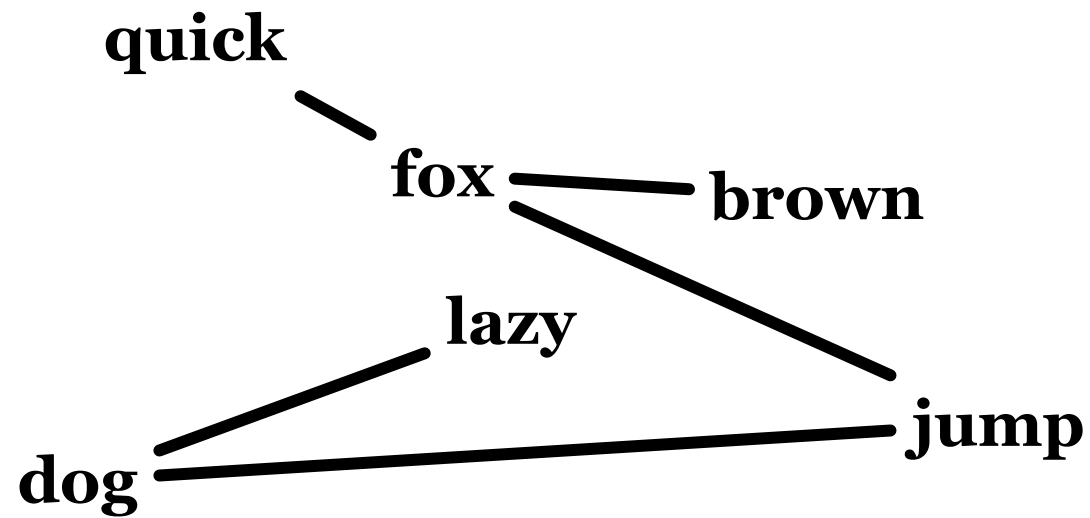
**brown**

**lazy**

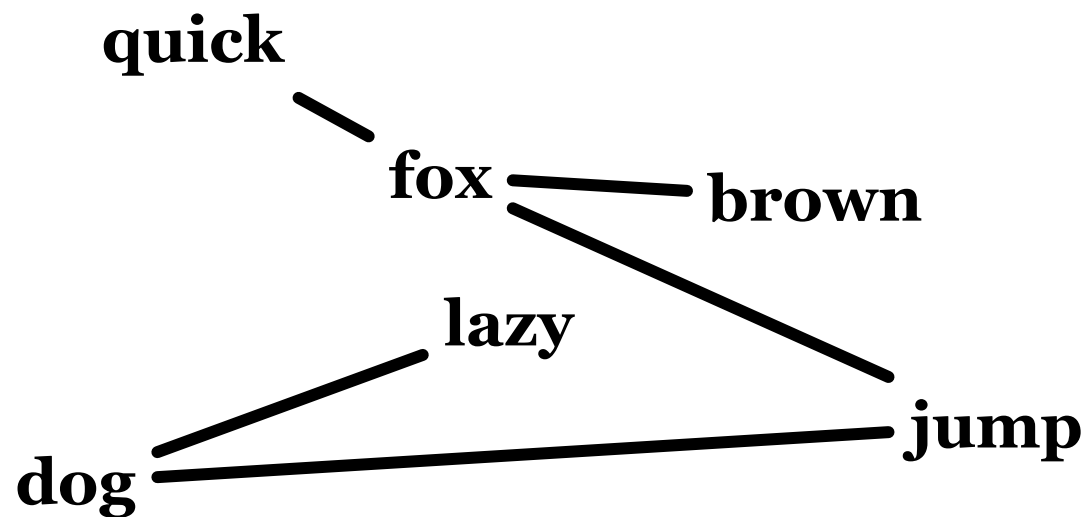
**dog**

**jump**

# Relationships



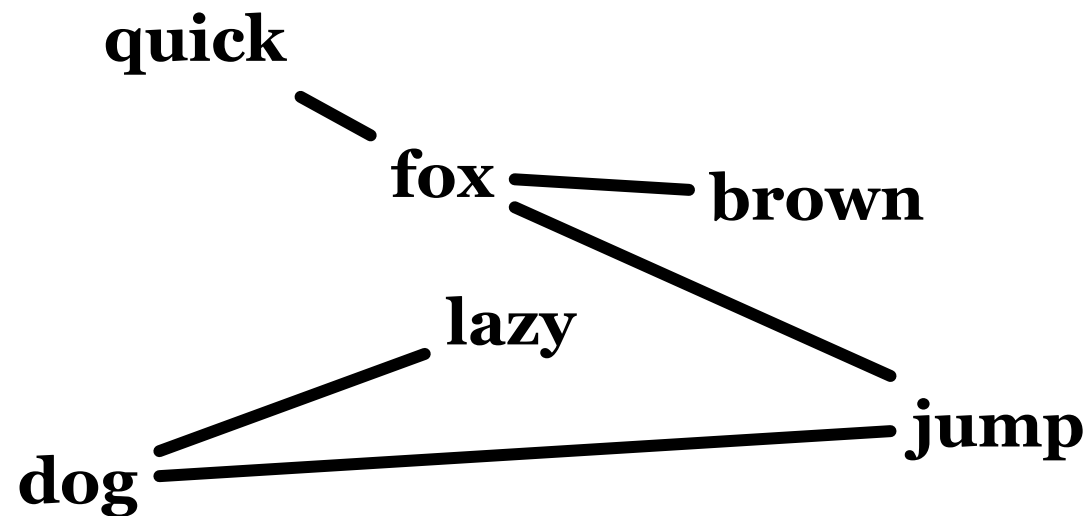
# Marking of Relationships: Word Order



quick brown fox jump lazy dog

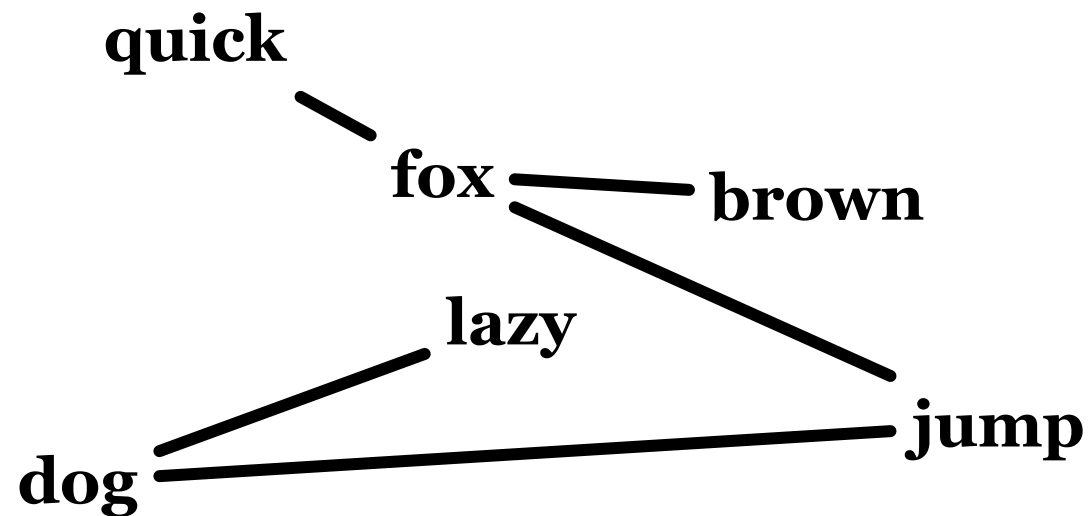


# Marking of Relationships: Function Words



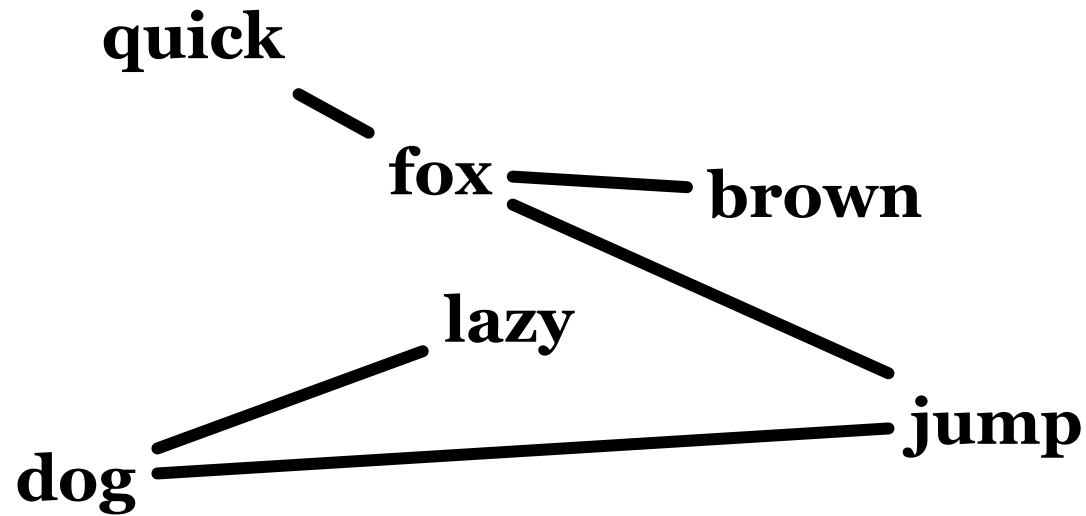
quick brown fox jump **over** lazy dog

# Marking of Relationships: Morphology



quick brown fox jump<sup>s</sup> over lazy dog

# Some Nuance

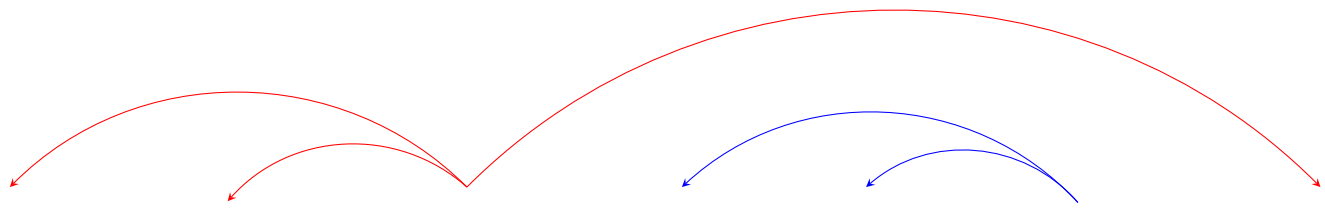


the quick brown fox jumps over the lazy dog

# Marking of Relationships: Agreement

- From Catullus, First Book, first verse (Latin):

Cui dono lepidum novum libellum arida modo pumice expoliturum ?  
Whom I-present lovely new little-book dry manner pumice polished ?  
(To whom do I present this lovely new little book now polished with a dry pumice?)



- Gender (and case) agreement links adjectives to nouns

# Marking of Relationships to Verb: Case

- German:

Die Frau	gibt	dem Mann	den Apfel
The woman	gives	the man	the apple
subject		indirect object	object

Der Frau	gibt	der Mann	den Apfel
The woman	gives	the man	the apple
indirect object		subject	object

- Case inflection indicates role of noun phrases

# Case Morphology vs. Prepositions

- Two different word orderings for English:
  - The woman gives the man the apple
  - The woman gives the apple **to** the man

- Japanese:

女性は	男性に	アップルの	を与えます
woman SUBJ	man OBJ	apple OBJ2	gives

- Is there a real difference between prepositions and noun phrase case inflection?

# Words

This is a simple sentence      **WORDS**

# Morphology

This is a simple sentence

be  
3sg  
present

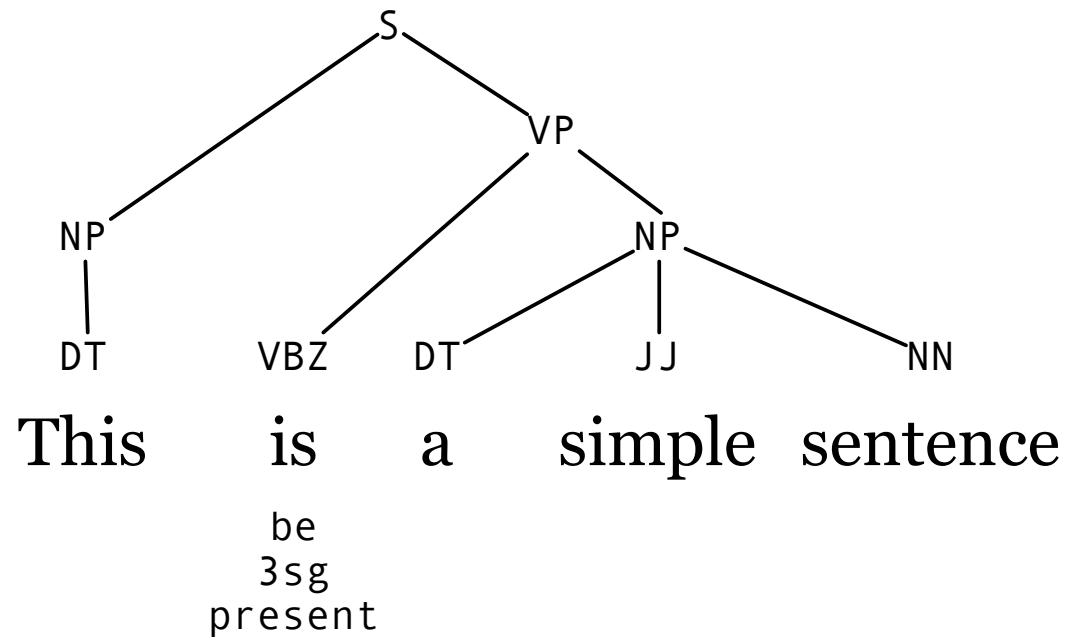
**WORDS**  
**MORPHOLOGY**



# Parts of Speech

DT	VBZ	DT	JJ	NN	<b>PART OF SPEECH</b>
This	is	a	simple	sentence	<b>WORDS</b>
	be 3sg present				<b>MORPHOLOGY</b>

# Syntax

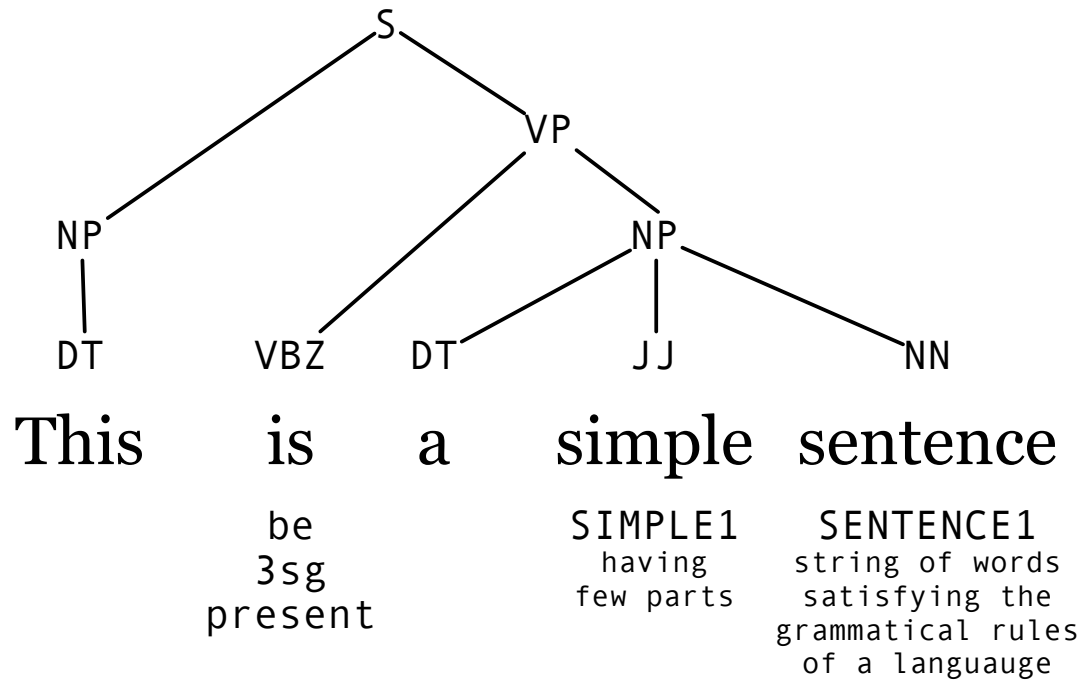


**SYNTAX**

**PART OF SPEECH**

**WORDS**

**MORPHOLOGY**



**SYNTAX**

**PART OF SPEECH**

**WORDS**

**MORPHOLOGY**

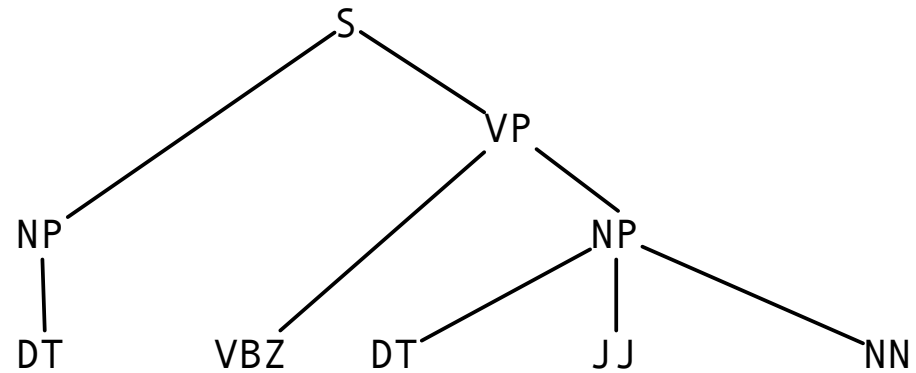
**SEMANTICS**

be  
3sg  
present

SIMPLE1  
having  
few parts

SENTENCE1  
string of words  
satisfying the  
grammatical rules  
of a language

# Discourse



**SYNTAX**

**PART OF SPEECH**

**This is a simple sentence**

**WORDS**

be  
3sg  
present

SIMPLE1  
having  
few parts

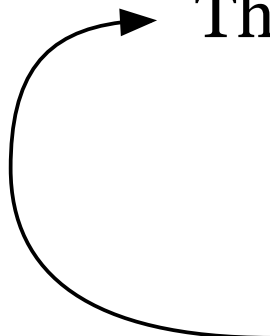
SENTENCE1  
string of words  
satisfying the  
grammatical rules  
of a language

**MORPHOLOGY**

**SEMANTICS**

**DISCOURSE**

CONTRAST



**But it is an instructive one.**

# Why is Language Hard?



- Ambiguities on many levels
- Rules, but many exceptions
- No clear understand how humans process language
- Can we learn everything about language by automatic data analysis?

data

- Definition: strings of letters separated by spaces
- But how about:
  - punctuation: commas, periods, etc. typically separated (tokenization)
  - hyphens: **high-risk**
  - clitics: **Joe's**
  - compounds: **website, Computerlinguistikvorlesung**
- And what if there are no spaces:

伦敦每日快报指出,两台记载黛安娜王妃一九九七年巴黎死亡车祸调查资料的手提电脑,被从前大都会警察总长的办公室里偷走.

# Word Counts

Most frequent words in the English Europarl corpus

any word		nouns	
Frequency in text	Token	Frequency in text	Content word
1,929,379	the	129,851	European
1,297,736	,	110,072	Mr
956,902	.	98,073	commission
901,174	of	71,111	president
841,661	to	67,518	parliament
684,869	and	64,620	union
582,592	in	58,506	report
452,491	that	57,490	council
424,895	is	54,079	states
424,552	a	49,965	member



# Word Counts

But also:

There is a large tail of words that occur only once.

33,447 words occur once, for instance

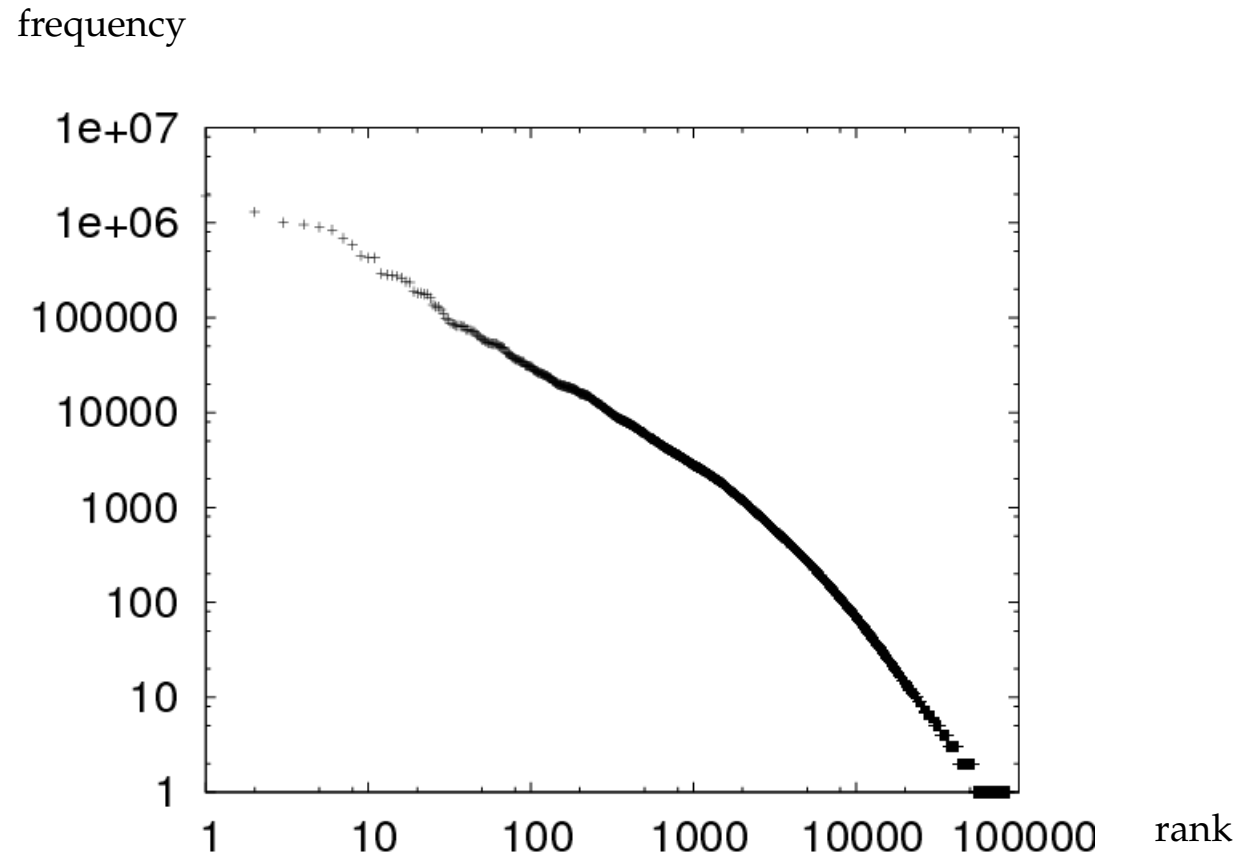
- [cornflakes](#)
- [mathematicians](#)
- [Tazhikhistan](#)

# Zipf's law

$$f \times r = k$$

$f$  = frequency of a word  
 $r$  = rank of a word (if sorted by frequency)  
 $k$  = a constant

# Zipf's law as a graph



Why a line in log-scale?

$$fr = k \Rightarrow f = \frac{k}{r} \Rightarrow \log f = \log k - \log r$$

# statistics

- Given word counts we can estimate a probability distribution:

$$P(w) = \frac{\text{count}(w)}{\sum_{w'} \text{count}(w')} \blacksquare$$

- This type of estimation is called *maximum likelihood estimation*. Why? We will get to that later.  $\blacksquare$
- Estimating probabilities based on frequencies is called the *frequentist approach* to probability.  $\blacksquare$
- This probability distribution answers the question: If we randomly pick a word out of a text, how likely will it be word  $w$ ?

# A Bit More Formal

- We introduce a **random variable**  $W$ .
- We define a **probability distribution**  $p$ , that tells us how likely the variable  $W$  is the word  $w$ :

$$\text{prob}(W = w) = p(w)$$

- Sometimes, we want to deal with two random variables at the same time.
- Example: Words  $w_1$  and  $w_2$  that occur in sequence (a **bigram**)  
We model this with the distribution:  $p(w_1, w_2)$
- If the occurrence of words in bigrams is **independent**, we can reduce this to  $p(w_1, w_2) = p(w_1)p(w_2)$ . Intuitively, this not the case for word bigrams.
- We can estimate **joint probabilities** over two variables the same way we estimated the probability distribution over a single variable:

$$p(w_1, w_2) = \frac{\text{count}(w_1, w_2)}{\sum_{w_1', w_2'} \text{count}(w_1', w_2')}$$

- Another useful concept is **conditional probability**

$$p(w_2|w_1)$$

It answers the question: If the random variable  $W_1 = w_1$ , how what is the value for the second random variable  $W_2$ ?

- Mathematically, we can define conditional probability as

$$p(w_2|w_1) = \frac{p(w_1, w_2)}{p(w_1)}$$

- If  $W_1$  and  $W_2$  are independent:  $p(w_2|w_1) = p(w_2)$



- A bit of math gives us the chain rule:

$$p(w_2|w_1) = \frac{p(w_1, w_2)}{p(w_1)}$$

$$p(w_1) p(w_2|w_1) = p(w_1, w_2)$$

- What if we want to break down large joint probabilities like  $p(w_1, w_2, w_3)$ ?

We can repeatedly apply the chain rule:

$$p(w_1, w_2, w_3) = p(w_1) p(w_2|w_1) p(w_3|w_1, w_2)$$

# Bayes Rule

- Finally, another important rule: **Bayes rule**

$$p(x|y) = \frac{p(y|x) p(x)}{p(y)}$$

- It can easily derived from the chain rule:

$$p(x, y) = p(x, y)$$

$$p(x|y) p(y) = p(y|x) p(x)$$

$$p(x|y) = \frac{p(y|x) p(x)}{p(y)}$$

# Expectation

- We introduced the concept of a random variable  $X$

$$\text{prob}(X = x) = p(x)$$

- Example: Roll of a dice. There is a  $\frac{1}{6}$  chance that it will be 1, 2, 3, 4, 5, or 6.

- We define the **expectation**  $E(X)$  of a random variable as:

$$E(X) = \sum_x p(x) x$$

- Roll of a dice:

$$E(X) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5$$

- **Variance** is defined as

$$\text{Var}(X) = E((X - E(X))^2) = E(X^2) - E^2(X)$$

$$\text{Var}(X) = \sum_x p(x) (x - E(X))^2$$

- Intuitively, this is a measure how far events diverge from the mean (expectation)
- Related to this is **standard deviation**, denoted as  $\sigma$ .

$$\text{Var}(X) = \sigma^2$$

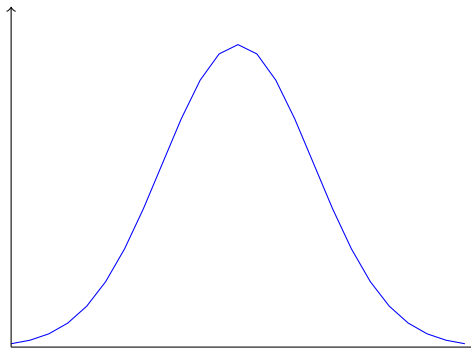
$$E(X) = \mu$$

- Roll of a dice:

$$\begin{aligned} \text{Var}(X) &= \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2 \\ &\quad + \frac{1}{6}(4 - 3.5)^2 + \frac{1}{6}(5 - 3.5)^2 + \frac{1}{6}(6 - 3.5)^2 \\ &= \frac{1}{6}((-2.5)^2 + (-1.5)^2 + (-0.5)^2 + 0.5^2 + 1.5^2 + 2.5^2) \\ &= \frac{1}{6}(6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25) \\ &= 2.917 \end{aligned}$$

- **Uniform:** all events equally likely
  - $\forall x, y : p(x) = p(y)$
  - example: roll of one dice
  
- **Binomial:** a series of trials with only two outcomes
  - probability  $p$  for each trial, occurrence  $r$  out of  $n$  times:  
$$b(r; n, p) = \binom{n}{r} p^r (1 - p)^{n-r}$$
  - a number of coin tosses

- **Normal:** common distribution for continuous values
  - value in the range  $[-\infty, x]$ , given expectation  $\mu$  and standard deviation  $\sigma$ :
$$n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$
  - also called **Bell curve**, or **Gaussian**
  - examples: heights of people, IQ of people, tree heights, ...



- We introduced previously an estimation of probabilities based on frequencies:

$$P(w) = \frac{\text{count}(w)}{\sum_{w'} \text{count}(w')}$$

- Alternative view: Bayesian: what is the most likely model given the data

$$p(M|D)$$

- Model and data are viewed as random variables
  - model  $M$  as random variable
  - data  $D$  as random variable



- Reformulation of  $p(M|D)$  using Bayes rule:

$$p(M|D) = \frac{p(D|M) p(M)}{p(D)}$$

$$\operatorname{argmax}_M p(M|D) = \operatorname{argmax}_M p(D|M) p(M)$$

- $p(M|D)$  answers the question: What is the most likely model given the data
- $p(M)$  is a prior that prefers certain models (e.g. simple models)
- The frequentist estimation of word probabilities  $p(w)$  is the same as Bayesian estimation with a uniform prior (no bias towards a specific model), hence it is also called the **maximum likelihood estimation**

# Entropy

- An important concept is **entropy**:

$$H(X) = \sum_x -p(x) \log_2 p(x)$$

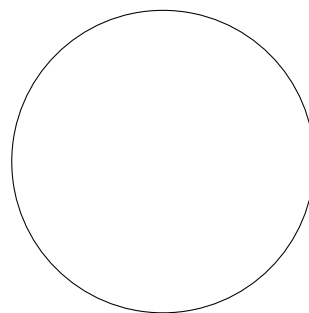
- A measure for the degree of disorder

# Entropy Example

One event

$$p(a) = 1$$

$$\begin{aligned} H(X) &= -1 \log_2 1 \\ &= 0 \end{aligned}$$

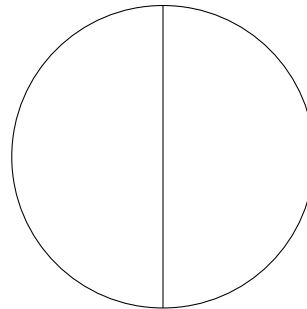


# Entropy Example

2 equally likely events:

$$p(a) = 0.5$$
$$p(b) = 0.5$$

$$H(X) = -0.5 \log_2 0.5 - 0.5 \log_2 0.5$$
$$= -\log_2 0.5$$
$$= 1$$

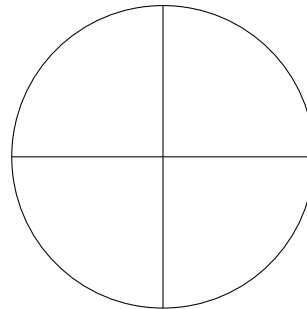


# Entropy Example

4 equally likely events:

$$\begin{aligned}p(a) &= 0.25 \\p(b) &= 0.25 \\p(c) &= 0.25 \\p(d) &= 0.25\end{aligned}$$

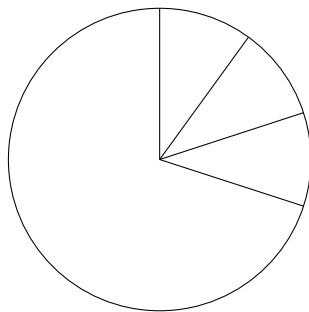
$$\begin{aligned}H(X) &= -0.25 \log_2 0.25 - 0.25 \log_2 0.25 \\&\quad - 0.25 \log_2 0.25 - 0.25 \log_2 0.25 \\&= -\log_2 0.25 \\&= 2\end{aligned}$$



# Entropy Example

4 events, one more likely than the others:

$$\begin{aligned}p(a) &= 0.7 \\p(b) &= 0.1 \\p(c) &= 0.1 \\p(d) &= 0.1\end{aligned}$$



$$\begin{aligned}H(X) &= -0.7 \log_2 0.7 - 0.1 \log_2 0.1 \\&\quad - 0.1 \log_2 0.1 - 0.1 \log_2 0.1 \\&= -0.7 \log_2 0.7 - 0.3 \log_2 0.1 \\&= -0.7 \times -0.5146 - 0.3 \times -3.3219 \\&= 0.36020 + 0.99658 \\&= 1.35678\end{aligned}$$

# Entropy Example

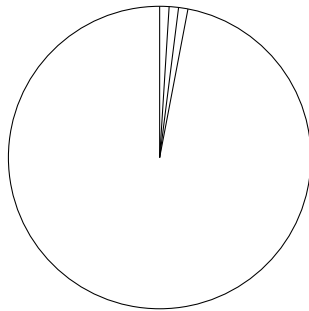
4 events, one much more likely than the others:

$$p(a) = 0.97$$

$$p(b) = 0.01$$

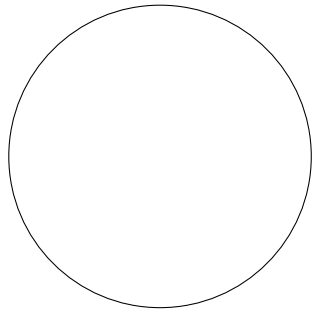
$$p(c) = 0.01$$

$$p(d) = 0.01$$

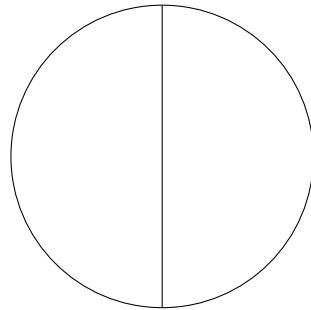


$$\begin{aligned} H(X) &= -0.97 \log_2 0.97 - 0.01 \log_2 0.01 \\ &\quad - 0.01 \log_2 0.01 - 0.01 \log_2 0.01 \\ &= -0.97 \log_2 0.97 - 0.03 \log_2 0.01 \\ &= -0.97 \times -0.04394 - 0.03 \times -6.6439 \\ &= 0.04262 + 0.19932 \\ &= 0.24194 \end{aligned}$$

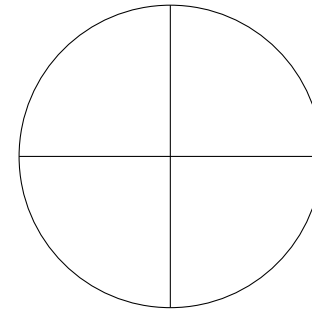
# Examples



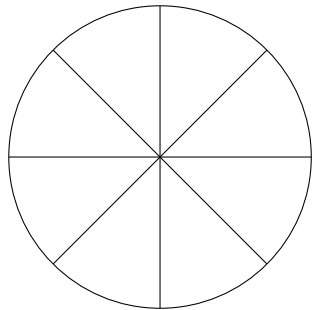
$$H(X) = 0$$



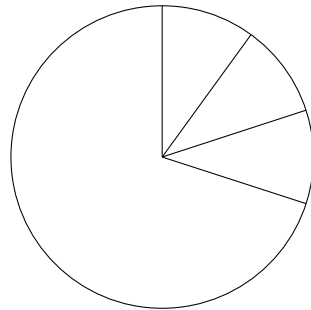
$$H(X) = 1$$



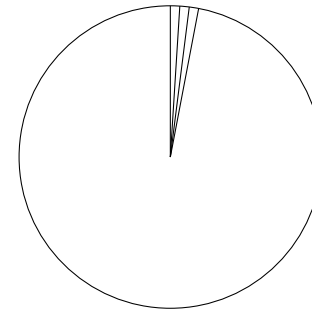
$$H(X) = 2$$



$$H(X) = 3$$



$$H(X) = 1.35678$$



$$H(X) = 0.24194$$



# Intuition Behind Entropy

- A good model has low entropy

→ it is more certain about outcomes

- For instance a translation table

$e$	$f$	$p(e f)$
the	der	0.8
that	der	0.2

is better than

$e$	$f$	$p(e f)$
the	der	0.02
that	der	0.01
...	...	...

- A lot of statistical estimation is about reducing entropy

- Assume that we want to encode a sequence of events  $X$
- Each event is encoded by a sequence of bits
- For example
  - Coin flip: heads = 0, tails = 1
  - 4 equally likely events: a = 00, b = 01, c = 10, d = 11
  - 3 events, one more likely than others: a = 0, b = 10, c = 11
  - Morse code: e has shorter code than  $q$
- Average number of bits needed to encode  $X \geq$  entropy of  $X$

# The Entropy of English

- We already talked about the probability of a word  $p(w)$
- But words come in sequence. Given a number of words in a text, can we guess the next word  $p(w_n | w_1, \dots, w_{n-1})$ ?
- Assuming a model with a limited window size

Model	Entropy
0th order	4.76
1st order	4.03
2nd order	2.8
human, unlimited	1.3

# Next Lecture: Language Models

- Next time, we will expand on the idea of a model of English in the form

$$p(w_n | w_1, \dots, w_{n-1})$$

- Despite its simplicity, a tremendously useful tool for NLP
- Nice machine learning challenge
  - sparse data
  - smoothing
  - back-off and interpolation