Game theory 2

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Sum-up of the previous lecture

Opponent

	s_1	s_3
S_2	0 , 3	6 , 4
S_3	1 , 5	5 , 2

Me

Social welfare

Social welfare

- Situation where sum of all utilities of an outcome is at its maximum

- Might provide unstable situations

- Does not provide a solid analytical tool

Game M

B

	Right	Left
Right	2 , 2	0 , 0
Left	0 , 0	1 , 1

A

Game M – Social welfare

B

	Right	Left
Right	4	0
Left	0	2

A

Game N

	B	
A	**l**	**r**
L	2 , 2	4 , 0
R	2 , 3	8 , -1

Game N – Social welfare

B

	l	r
L	4	4
R	5	**7**

A

Prisoner's dilemma – Social welfare

B

	c	d
C	5 , 5	0 , 7
D	7 , 0	3 , 3

A

Prisoner's dilemma – Social welfare

B

	c	d
C	10	7
D	7	6

A

Prisoner's dilemma – Social welfare

Prisoner's dilemma – Social welfare

B

	c	d
C	55	70
D	70	33

A

Prisoner's dilemma – Social welfare

B

		C	d
A	C	50 , 5	0 , 7
	D	70 , 0	30 , 3

Prisoner's dilemma – Social welfare

B

	c	D
C	55	7
D	**70**	33

A

Pareto optimality

Game M

B

	Right	Left
Right	2 , 2	0 , 0
Left	0 , 0	1 , 1

A

Game M

B

	Right	Left
Right	2 , 2	0 , 0
Left	0 , 0	1 , 1

A

Game M – pure strategy equilibriums

Pareto optimality

- Outcome is Pareto optimal (Pareto efficient), if there is **no other outcome** which is **better or equal for all players** and **strictly better for some player**

- To put it differently, there is no outcome which makes one player better without making other players worse – no outcome that Pareto dominates it

- Conversely, outcome A is Pareto dominated, if there is outcome B that makes all players as good (weakly better) and one player strictly better compared to outcome A

- Might provide unstable solutions
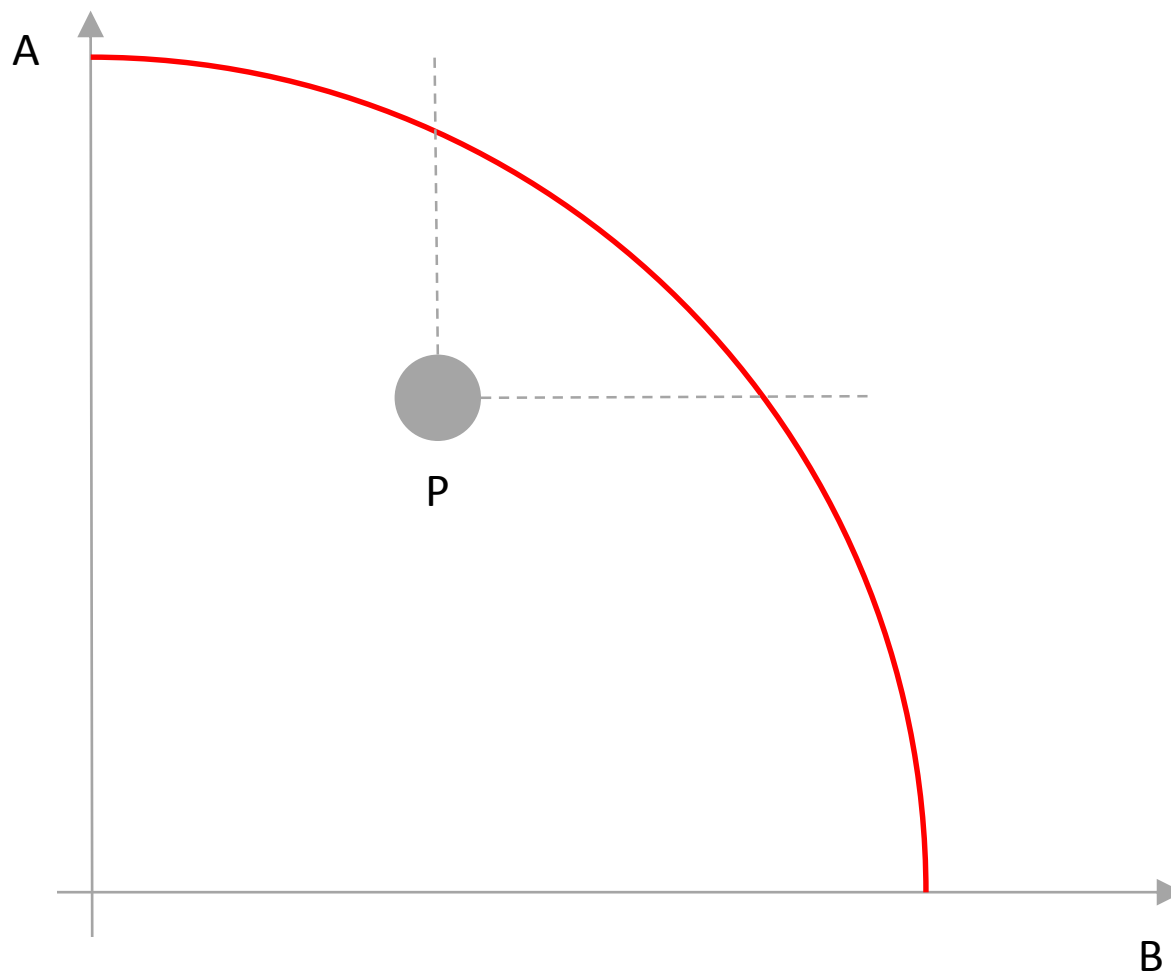
Game M – Pareto optimality

B

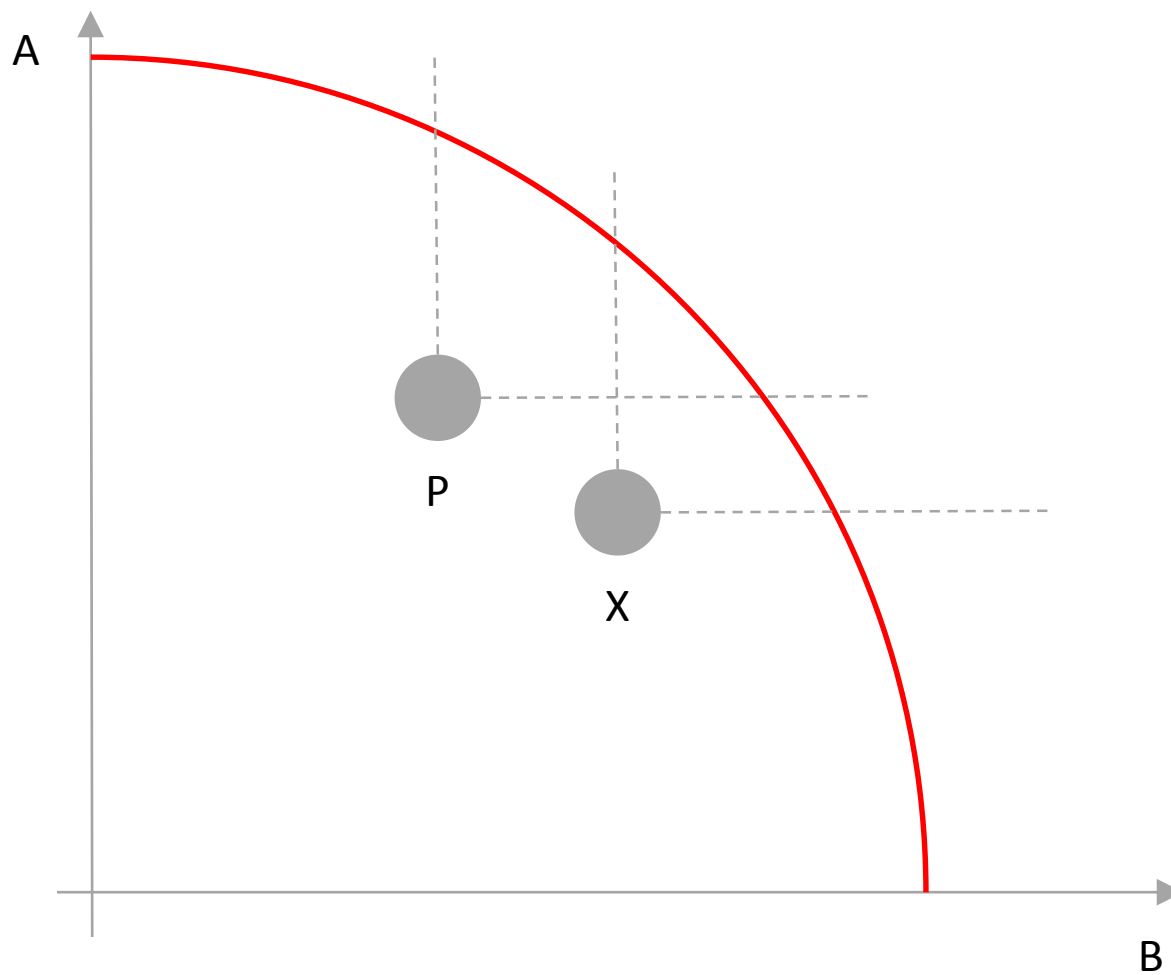|	Right	Left
Right	2 , 2	0 , 0
Left	0 , 0	1 , 1
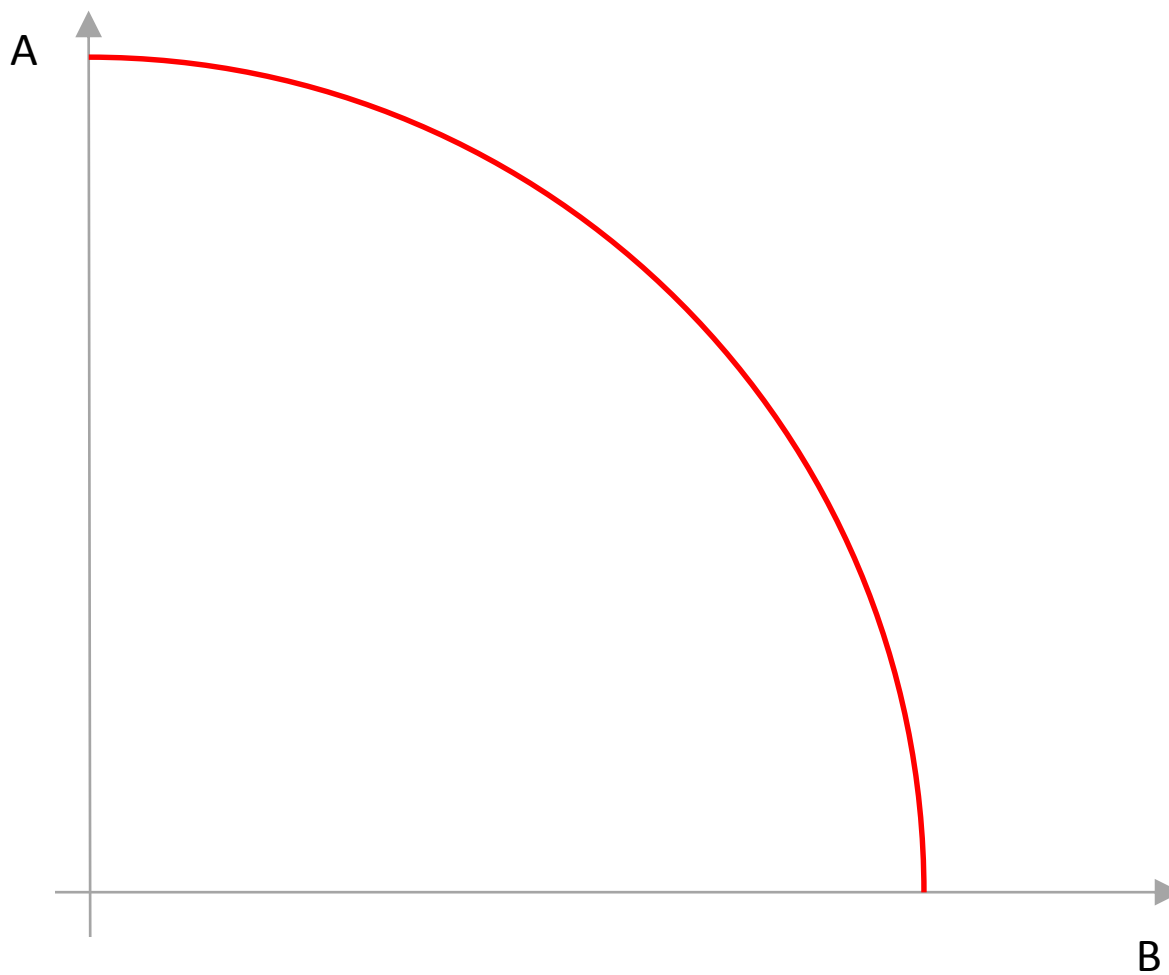
A

Pareto optimality

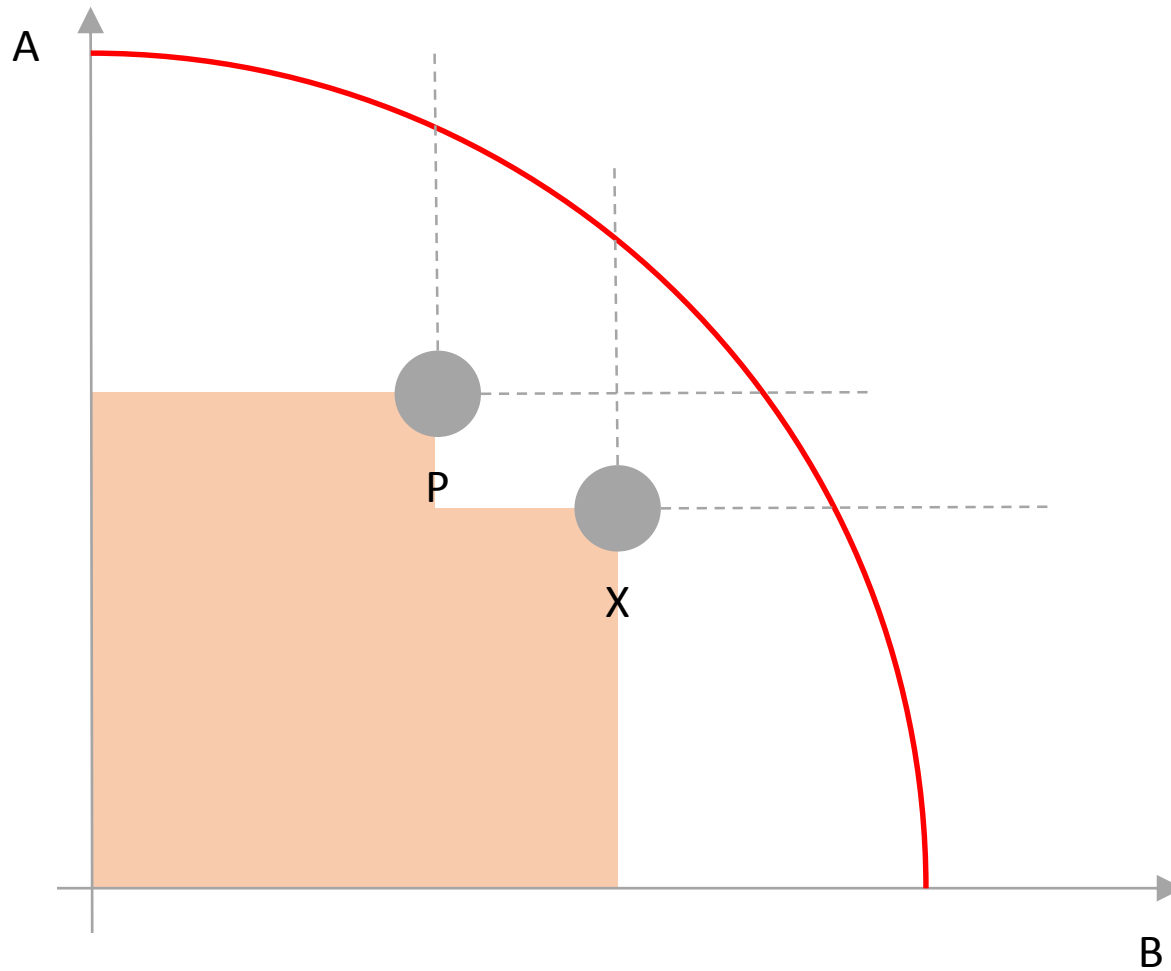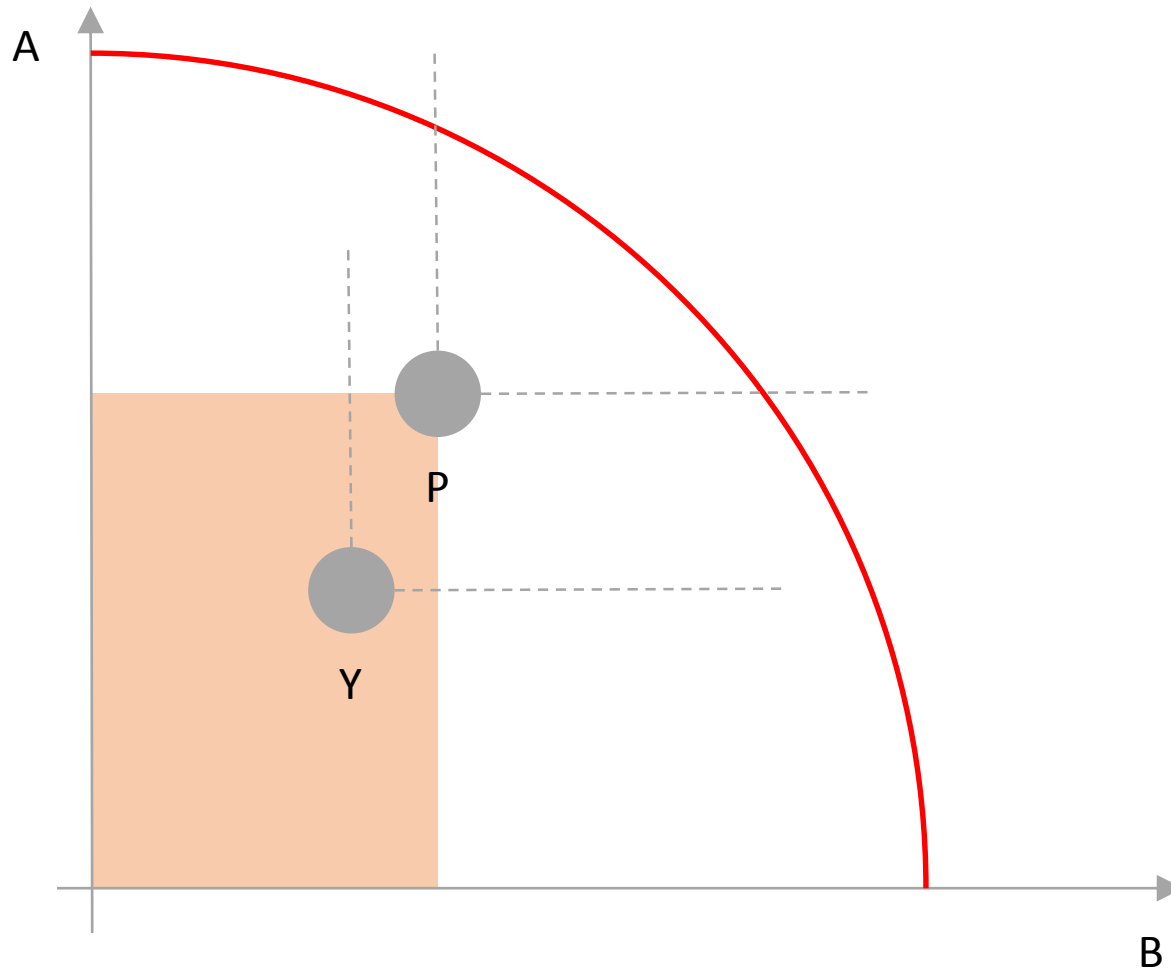# Pareto optimality

Pareto optimality

Pareto optimality
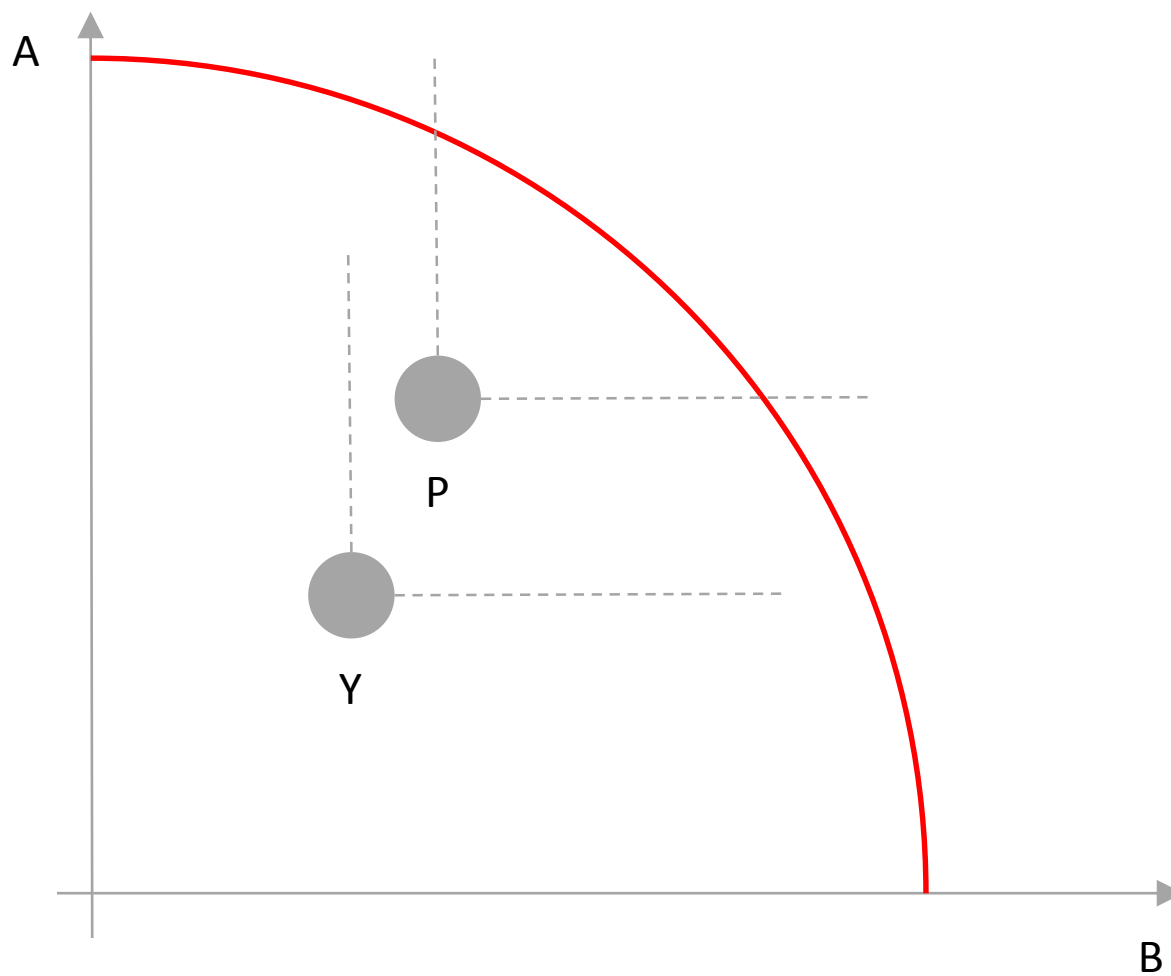
Pareto optimality

Pareto optimality

Prisoner's dilemma – Pareto optimality

		B	
		c	d
A	C	5 , 5	0 , 7
	D	7 , 0	3 , 3

Prisoner's dilemma – Pareto optimality

	c	d
C	5 , 5	0 , 7
D	7 , 0	3 , 3

B

A

Prisoner's dilemma – Pareto optimality

	c	d
C	5 , 5	0 , 7
D	7 , 0	3 , 3

A

B

Game N

		B	
		l	r
A	L	2 , 2	4 , 0
	R	2 , 3	8 , -1

Game N – Pareto optimality

Game N – Pareto optimality

Matching pennies – Pareto optimality

My pair

	Heads	Tails
Heads	1 , -1	-1 , 1
Tails	-1 , 1	1 , -1

Me

Matching pennies – Pareto optimality

My pair

	Heads	Tails
Heads	1 , -1	-1 , 1
Tails	-1 , 1	1 , -1

Me

Pareto optimality solid tool for comparing equilibriums

Mixed-strategy Nash equilibrium

Matching pennies

- Two players

- Players choose heads or tails

- If players match heads/tails, I (Player 1) win both coins

- If players don't match heads/tails, opponent (Player 2) wins both coins

Matching pennies

My pair

	Heads	Tails
Heads	1 , -1	-1 , 1
Tails	-1 , 1	1 , -1

Me

Matching pennies – mixed strategy

My pair

	Heads (0.5)	Tails (0.5)
Heads (0.5)	1 , -1	-1 , 1
Tails (0.5)	-1 , 1	1 , -1

Me

Calculation
of mixed-strategy NE

Game Y

	B: L	B: R
A: U	3 , -3	-2 , 2
A: D	-1 , 1	0 , 0

Game Y – Pareto efficiency

B

	L	R
U	3 , -3	-2 , 2
D	-1 , 1	0 , 0

A

Game Y – Pareto optimality?

Game Y

B

	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

A
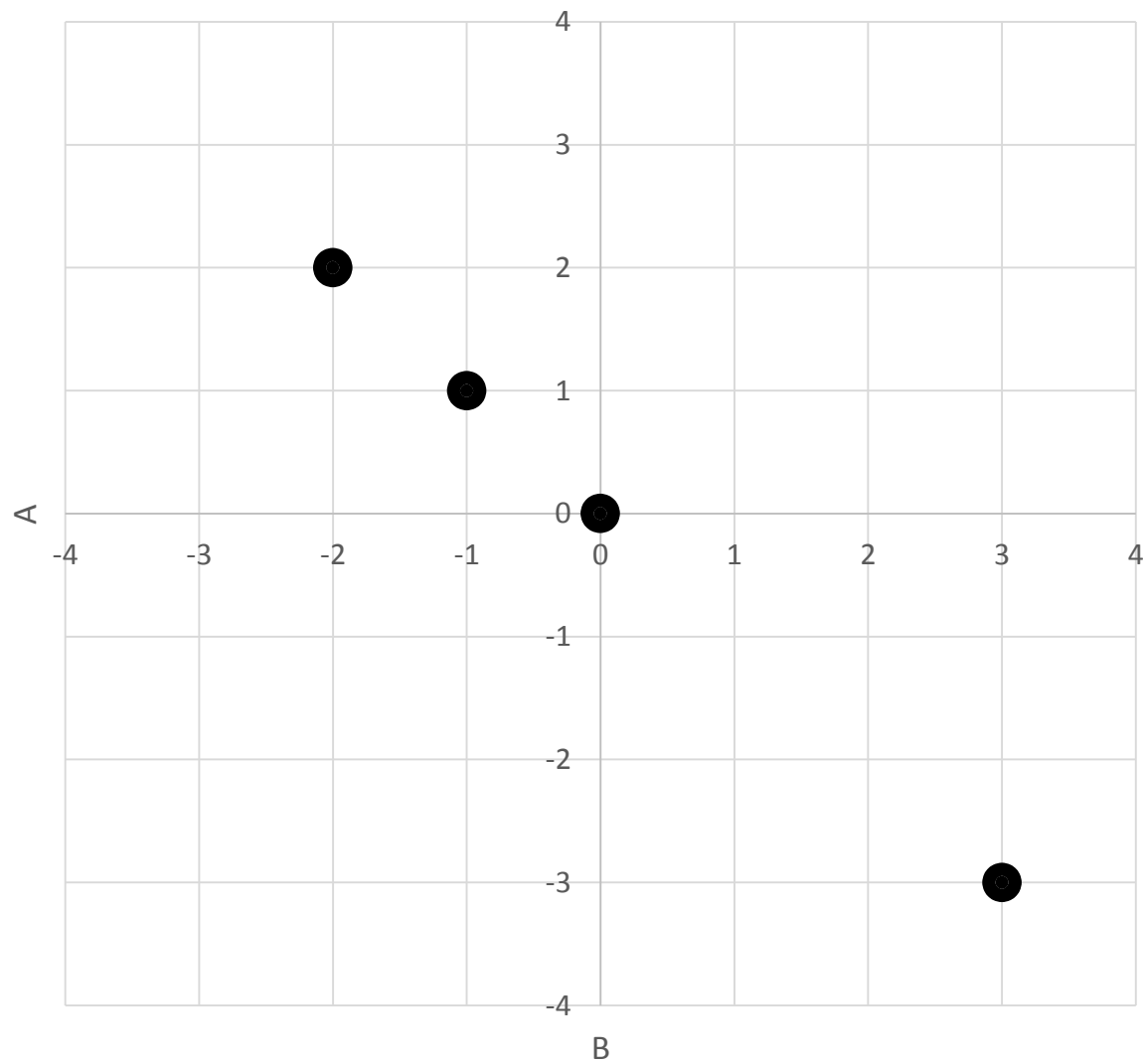
Game Y – Player A

- Player A plans to mix Up and Down strategy at a certain ratio p
- Player B might play Left or Right
- Player A must find such a probability of playing U and D that makes Player B indifferent to selecting L or R
- Player B has to gain same utility from B's choice Left and Right
  - $EU_L = EU_R$

- Expected utility of Player B chosing Left:
  - $EU_L = f(p)$

- Expected utility of Player B chosing Right:
  - $EU_R = f(p)$

Game Y - Player A's strategy

- EU_L:
  - Some % of time (p) gets B utility -3
  - Rest of the time (1 - p) gets B utility 1

- $EU_L = (p)*(-3) + (1 - p)*(1)$
- $EU_L = -3p + 1 - p$
- $EU_L = 1 - 4p$

B

	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

A

Game Y - Player A's strategy

- EU_R:
  - Some % of time (p) gets B utility 2
  - Rest of the time (1 - p) gets B utility 0

- $EU_R = (p)*(2) + (1 - p)*(0)$
- $EU_R = 2p + 0 - 0p$
- $EU_R = 2p$

B

	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

A

Player A's strategy – making B indifferent Comparison of EU_L with EU_R

- $EU_L = 1 - 4p$
- $EU_R = 2p$

- $EU_L \quad = EU_R$
- $1 - 4p \quad = 2p \qquad +4p$
- $1 \qquad = 6p \qquad /6$
- **p $\quad$ = 1/6**

- **1 - p** $= 1 - 1/6 =$ **5/6**

- We've found the ideal mixed strategy for Player A
- If Player A plays Up 1/6 of time and Down 5/6 of time, Player B is indifferent to choosing Left or Right
- We need to do the same for player B

Game Y - Player B's strategy

- EU_U:
  - Some % of time (q) gets A utility 3
  - Rest of the time (1 - q) gets A utility -2

- $EU_U = (q)*(3) + (1 - q)*(-2)$
- $EU_U = 3q - 2 + 2q$
- $EU_U = 5q - 2$

B

	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

A

Game Y - Player B's strategy

- EU_D:
  - Some % of time (q) gets A utility -1
  - Rest of the time (1 - q) gets A utility 0

- $EU_D = (q)*(-1) + (1 - q)*(0)$
- $EU_D = -1q + 0 - 0q$
- $EU_D = -q$

B

	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

A

Player B's strategy – making A indifferent Comparison of EU_U with EU_D

- $EU_U = 5q - 2$
- $EU_D = -q$

- $EU_U \qquad = EU_D$
- $5q - 2 \qquad = -q \qquad\qquad -5q$
- $\quad -2 \qquad = -6q \qquad\quad /-6$
- **q** $\qquad$ **= 1/3**

- **1 - q** = 1 - 1/3 = **2/3**

- We've found the ideal mixed strategy for Player B
- If Player B plays Left 1/3 of time and Down 2/3 of time, Player A is indifferent to choosing Up or Down

Mixed strategy NE

(1/6 U , 1/3 L)

Game Y - MSNE

B

	L (1/3)	R (2/3)
U (1/6)	3 , -3	-2 , 2
D (5/6)	-1 , 1	0 , 0

A

Battle of sexes

- Want to go out together but have no means of communication
- Have 2 choices – ballet or car show
- Player A prefers car show (C)
- Player B prefers ballet (B)
- Both prefer being together than being alone (A)

- Preferences for player A: C > B > A
- Preferences for player B: B > C > A

Battle of sexes

	b	c
B	1 , 2	0 , 0
C	0 , 0	2 , 1

A

B

Battle of sexes – PS Nash equilibriums

B

	b	c
B	1, 2	0, 0
C	0, 0	2, 1

A

Equilibriums

- 2 pure-strategies equilibriums

- How would they coordinate?

- Apart from pure strategies equilibriums there is one mixed strategy equilibrium for this game
- (1/3 B, 2/3 b)

Battle of sexes – mixed strategy equilibrium

B

	b 2/3	c 1/3
B 1/3	1 , 2	0 , 0
C 2/3	0 , 0	2 , 1

A

Calculation of MS NE payoffs

Battle of sexes – mixed-strategy NE payoffs

B

	b 2/3	c 1/3
B **1/3**	1 , 2 1/3 * 2/3	0 , 0 1/3 * 1/3
C **2/3**	0 , 0 2/3 * 2/3	2 , 1 2/3 * 1/3

A

Battle of sexes – mixed-strategy NE payoffs

B

	b 2/3	c 1/3
B 1/3	1 , 2 2/9	0 , 0 1/9
C 2/3	0 , 0 4/9	2 , 1 2/9

A

BoS – Payoffs for player A

- We simply multiply payoffs for player A and probabilities for each outcome and then sum them together

- Player A's payoffs:
  - u(B, b)   = 1 * 2/9   = 2/9
  - u(B, c)   = 0 * 1/9   = 0
  - u(C, b)   = 0 * 4/9   = 0
  - u(C, c)   = 2 * 2/9   = 4/9

- EU(A) = 2/9 + 0 + 0 + 4/9
- EU(A) = 6/9
- EU(A) = 2/3

BoS – Payoffs for player B

- We simply multiply payoffs of player B and probabilities for each outcome and then sum them together

- Player A's payoffs:
    - u(B, b) = 2 * 2/9   = 4/9
    - u(B, c) = 0 * 1/9   = 0
    - u(C, b) = 0 * 4/9   = 0
    - u(C, c) = 1 * 2/9   = 2/9

- EU(B) = 4/9 + 0 + 0 + 2/9
- EU(B) = 6/9
- EU(B) = 2/3

B

	b 2/3	c 1/3
B 1/3	1 , 2 2/9	0 , 0 1/9
C 2/3	0 , 0 4/9	2 , 1 2/9

A

Battle of sexes NE

- Pure strategies NE
  - (B , b)
    - EU(A) = 1
    - EU(B) = 2
  - (C , c)
    - EU(A) = 2
    - EU(B) = 1
- Mixed strategies NE
  - (1/3 B , 2/3 b)
    - EU(A) = 2/3
    - EU(B) = 2/3

B

	b	c
B	1 , 2	0 , 0
C	0 , 0	2 , 1

A

FSS entrance game

- Two students meet at the main faculty entrance
- Both simultaneously decide whether to walk or stop
- If both walk, they collide and both get a bruise (payoff -5)
- If one stops and other walks
  - Student who stopped gets good karma for letting the other pass with payoff 1, but at the same time gets delayed, which is completely offsetting the value of the good karma
  - Student who walked gets to pass quickly and thus gets payoff 1
- If both stop, each would get good karma for letting the other pass, but each will get delayed

FSS entrance game

B

		W	s
		W	s
A	W	-5 , -5	1 , 0
	S	0 , 1	0 , 0

FSS entrance game NE

Extensive form games

Extensive form games

- Visualized as a game (decision) tree

- Players move sequentially

- Captures time in game

- Captures knowledge of agents – sometimes agents do not have information where they are located in game
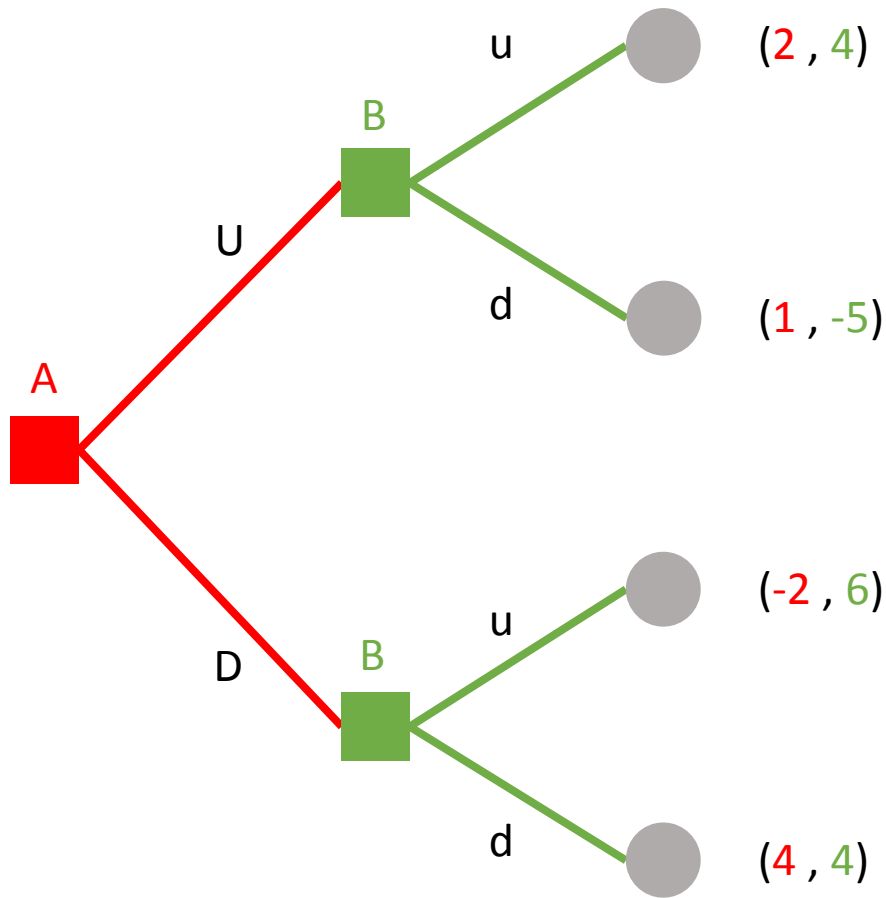
Basic terminology

- Each square is called node

- Each line represents an action an owner of the node has at his disposal

- Nodes might either trigger other action or end

- Every circle is an end of the tree – it's called terminal node

- Every circle must yield payoffs for all the actors

- Each moment actor has information about all previous moves called information set

Backwards induction
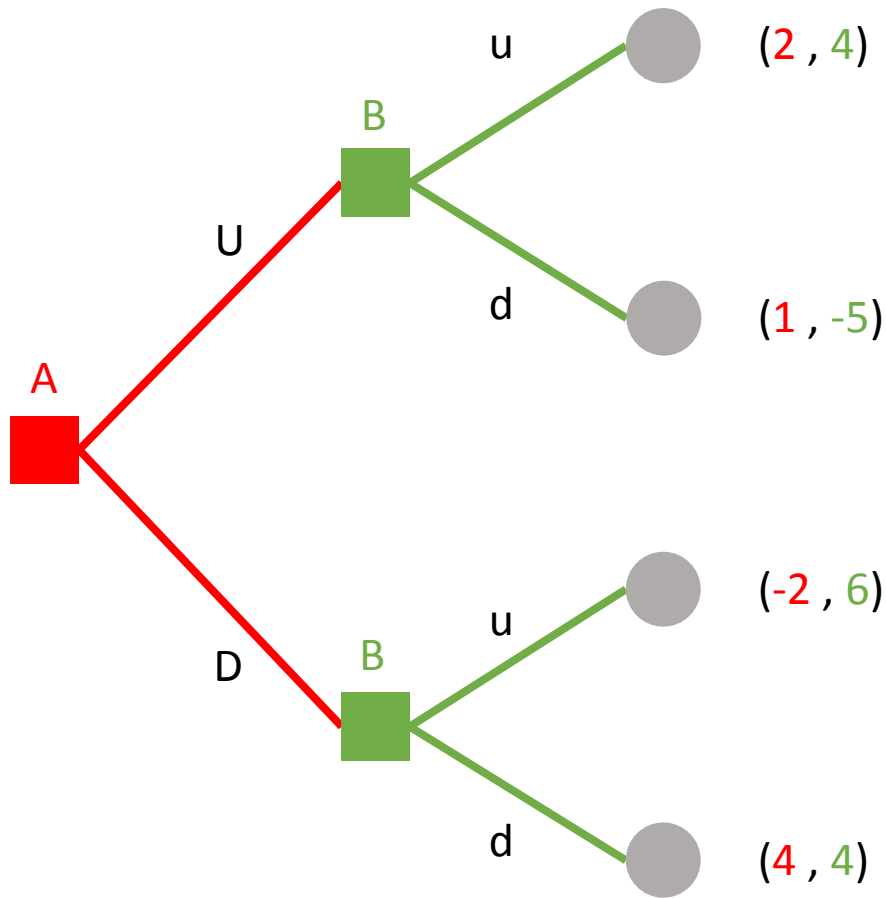
Backwards induction

- Moves player make at nodes reached in an equilibrium are called behavior on the equilibrium path

- Moves player make at nodes that are not reached in an equilibrium are behavior off the equilibrium path

- Players can not commit themselves to make moves in future – decisions at nodes change decisions at later nodes

Backwards induction

- Begin with decisions that lead only to terminal nodes

- Compare payoffs for decisions in each node leading to terminal node

- Find best reply to alternatives of player playing at the current node

- Work through nodes backwards and solve the outcomes of all nodes comparing payoffs for respective players

A

U

B

u

(2 , 4)

d

(1 , -5)

D

B

u

(-2 , 6)

d

(4 , 4)
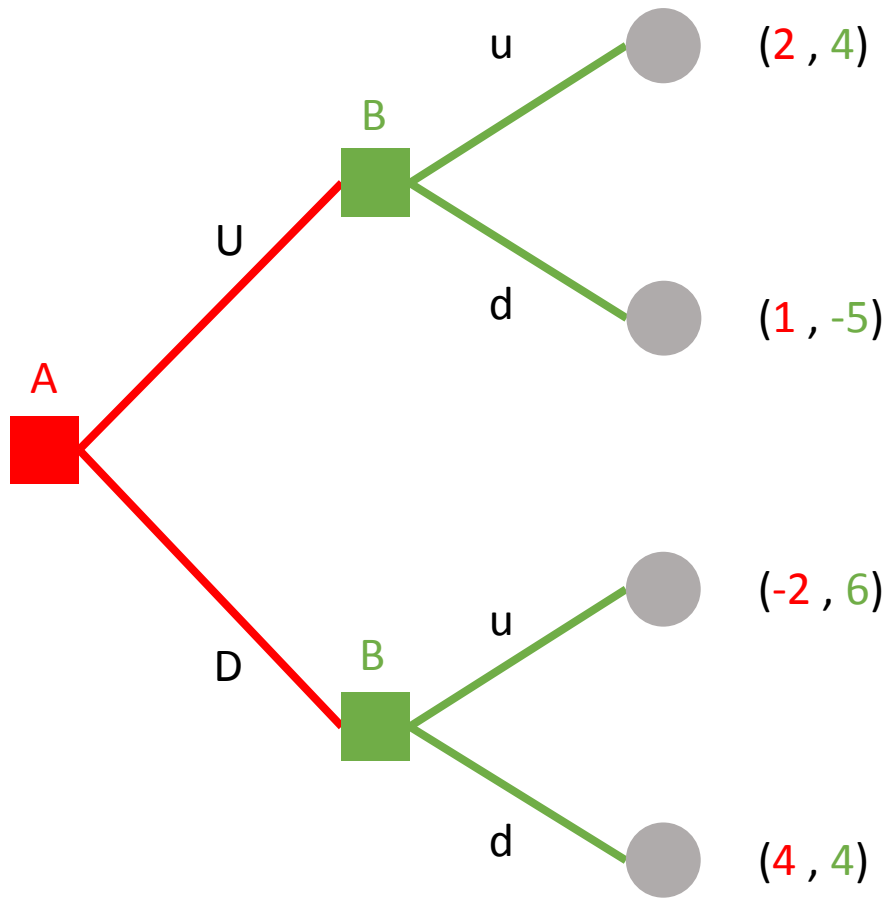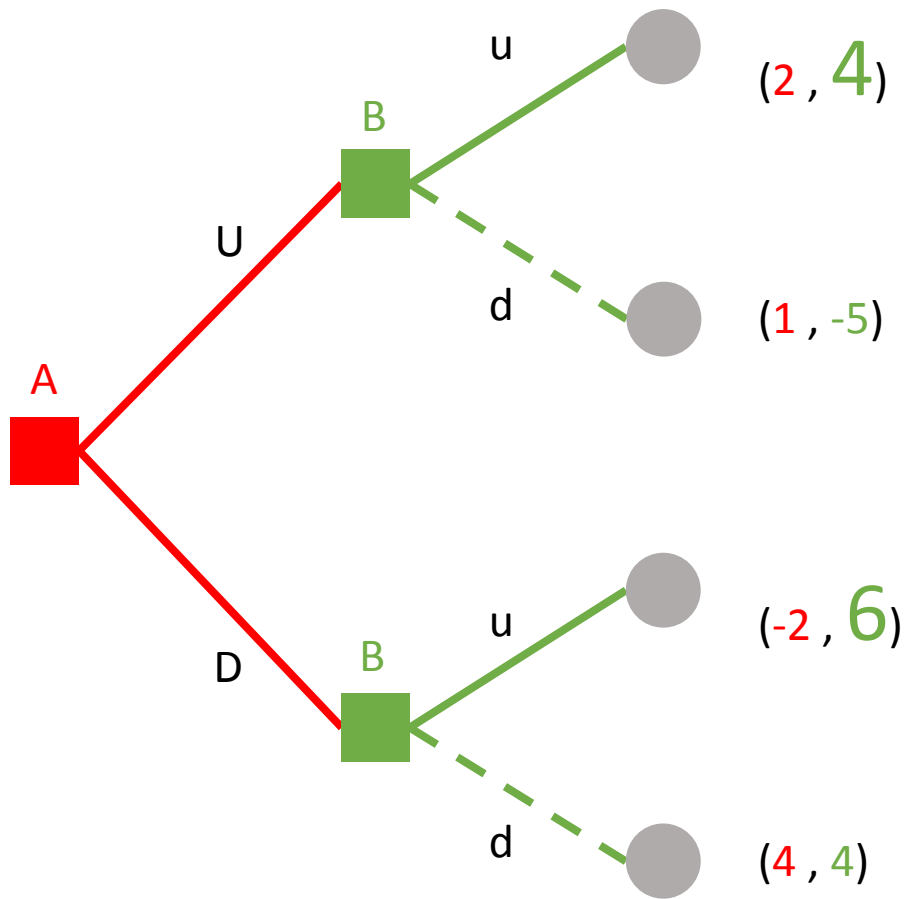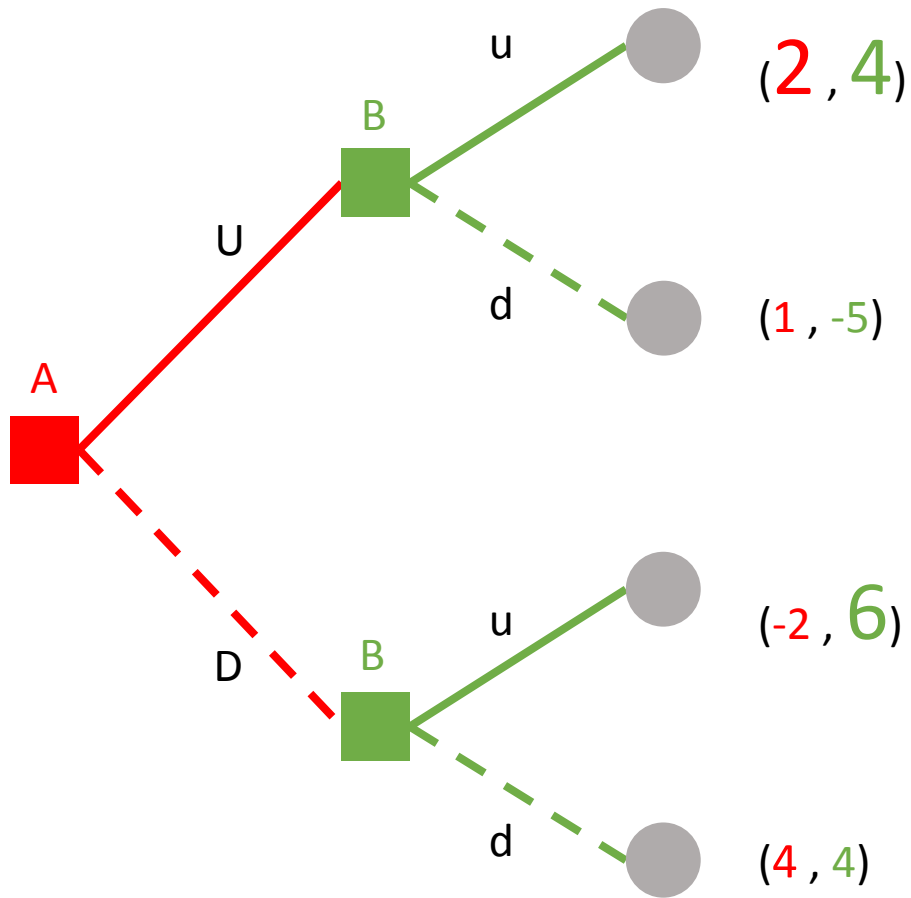
Equilibrium of sequential game

- One Nash equilibrium in pure strategies on the equilibrium path
- (U ; u , u)

- There may be **more NE in pure strategies**

- B decides – if A goes U than u yields better payoff in the upper node, if A goes D than u yields better payoff in the lower node

- A knows that B will choose u in both nodes, therefore compares payoffs in u for going U or D – U yields better payoff

Backwards induction

- Works easily in games of perfect information

- All actors are aware of all previous actions and can also anticipate, what actors will do based on their expected utilities over outcomes at subsequent nodes – actors have a perfect recall

- However, backwards induction assesses only rationality on the equilibrium path. NE found off the equilibrium path will not be found through backwards induction

Selten's game
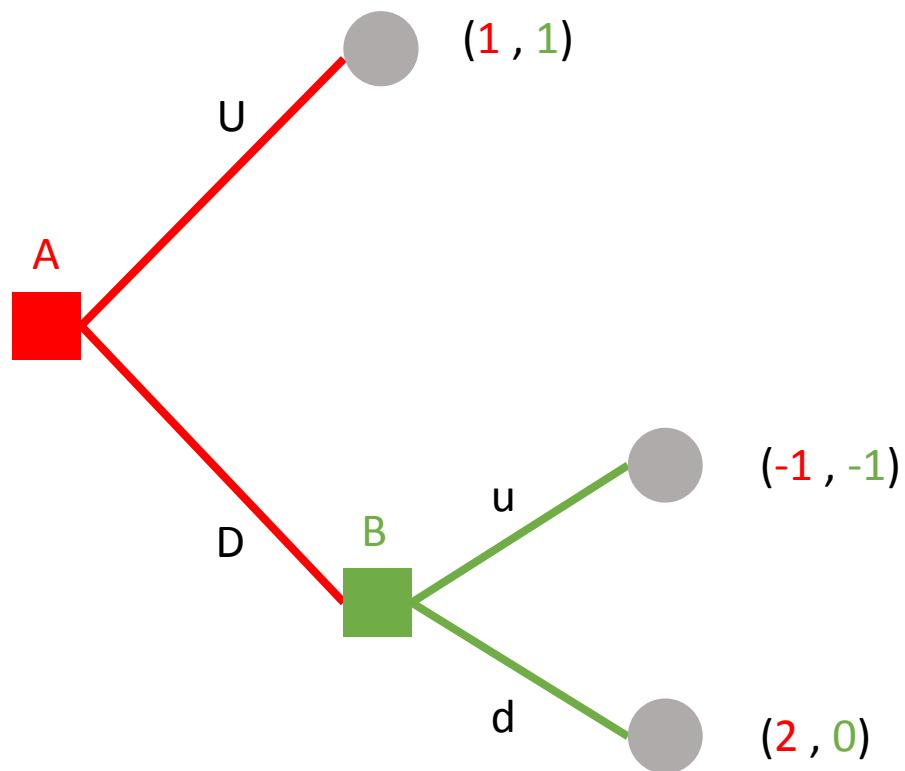
Selten's game
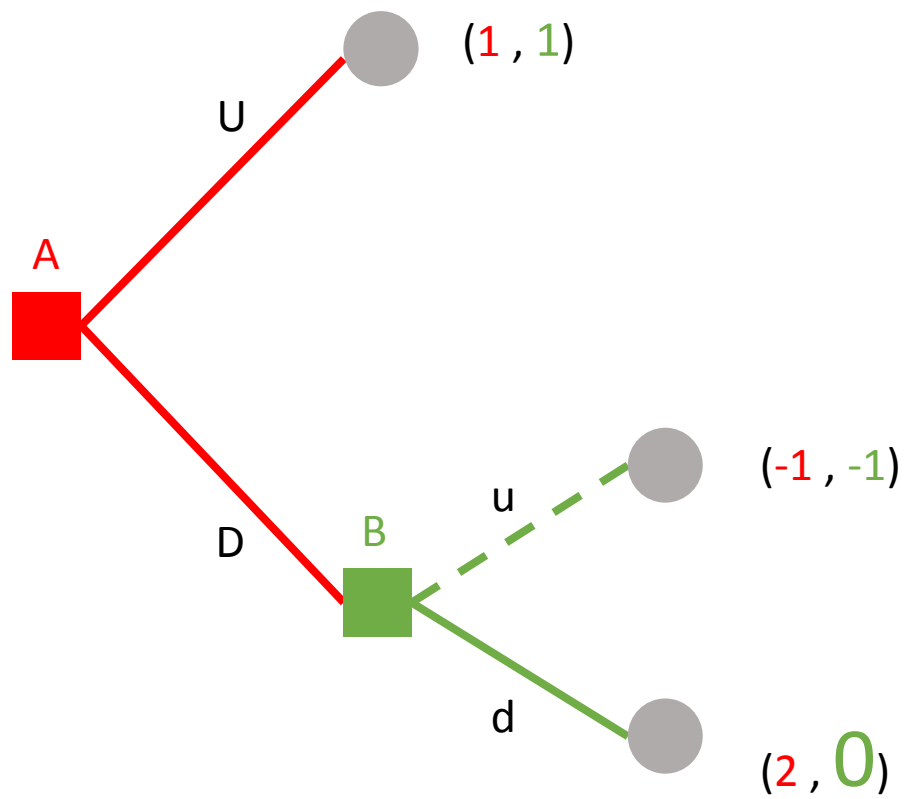
Selten's game
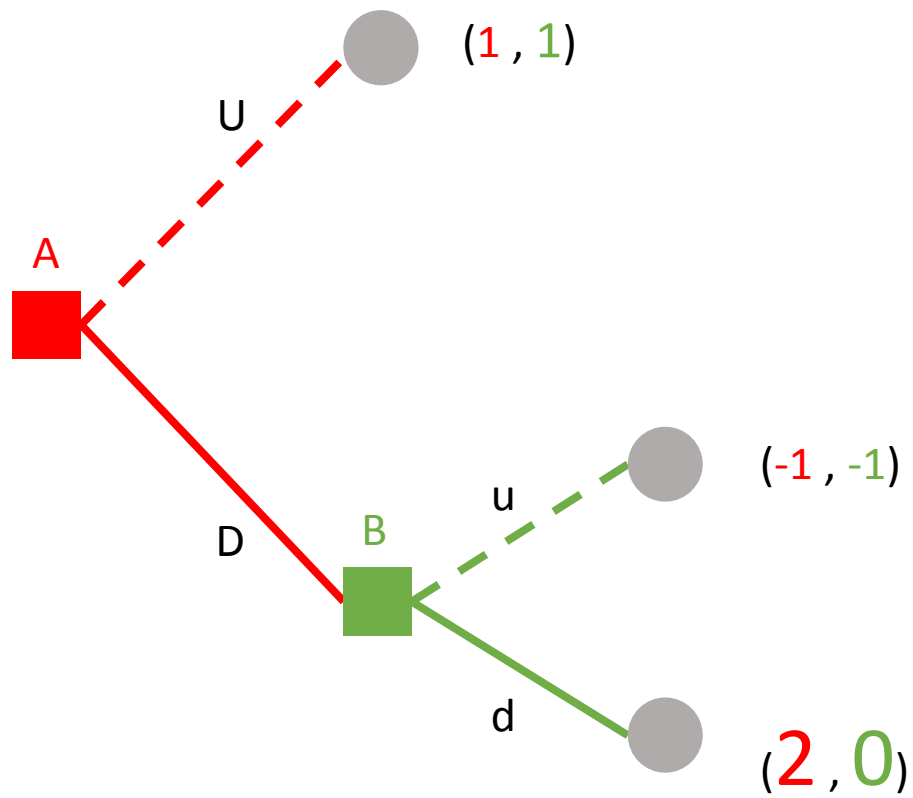
Rewrite Selten's game into matrix

- Player A has 2 moves
  - U, D
  - Represented as 2 rows in a matrix

- Player B has also 2 moves
  - u, d
  - Represented as 2 columns in a matrix

- We have 2x2 game in normal form

Selten's game

Strategic form of Selten's game
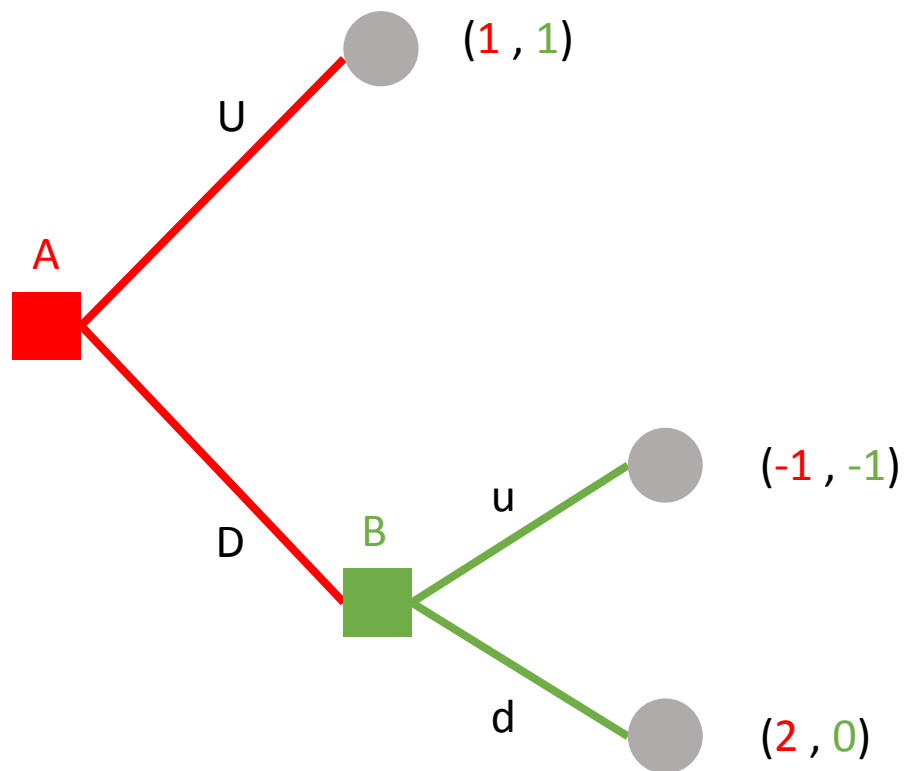
B

	u	d
U		
D		

A

Selten's game

Strategic form of Selten's game

B

	u	d
U		
D	-1 , -1	2 , 0

A

Selten's game

Strategic form of Selten's game

B

	u	d
U	1 , 1	
D	-1 , -1	2 , 0

A

Strategic form of Selten's game

B

	u	d
U	1 , 1	1 , 1
D	-1 , -1	2 , 0

A
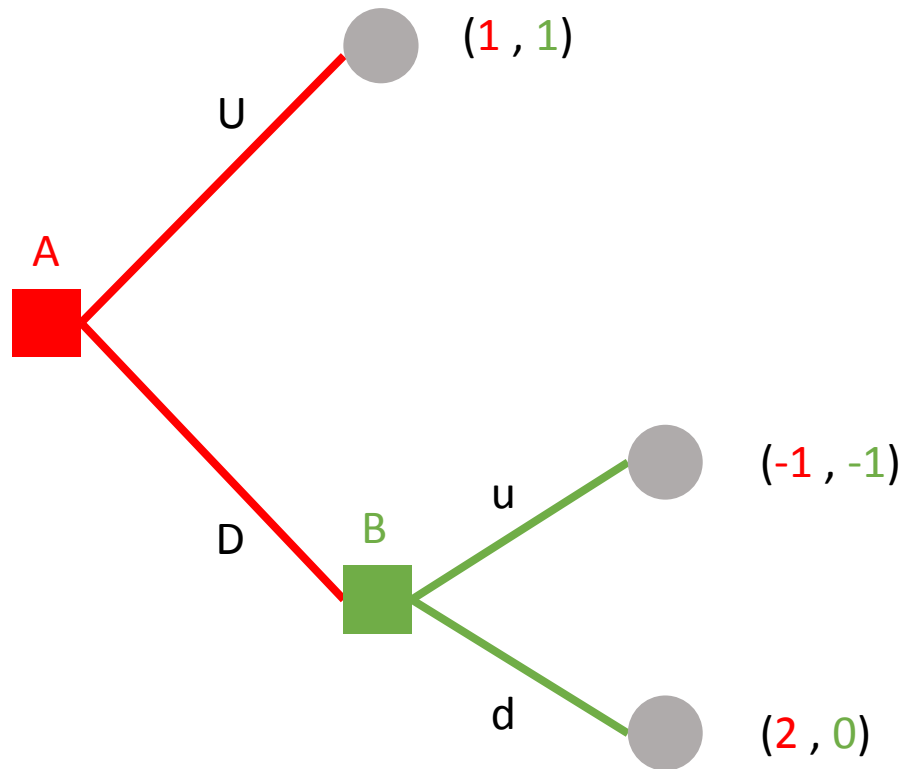
Rewriting extensive form into normal form

- If we rewrite extensive form game into normal form, only one matrix will emerge as a representation

- This does not work the other way around

- Since matrixes do not hold information about sequence of actions, one normal-form game might have multiple extensive-form representations that would completely alter outcomes of the game

Selten's game from matrix
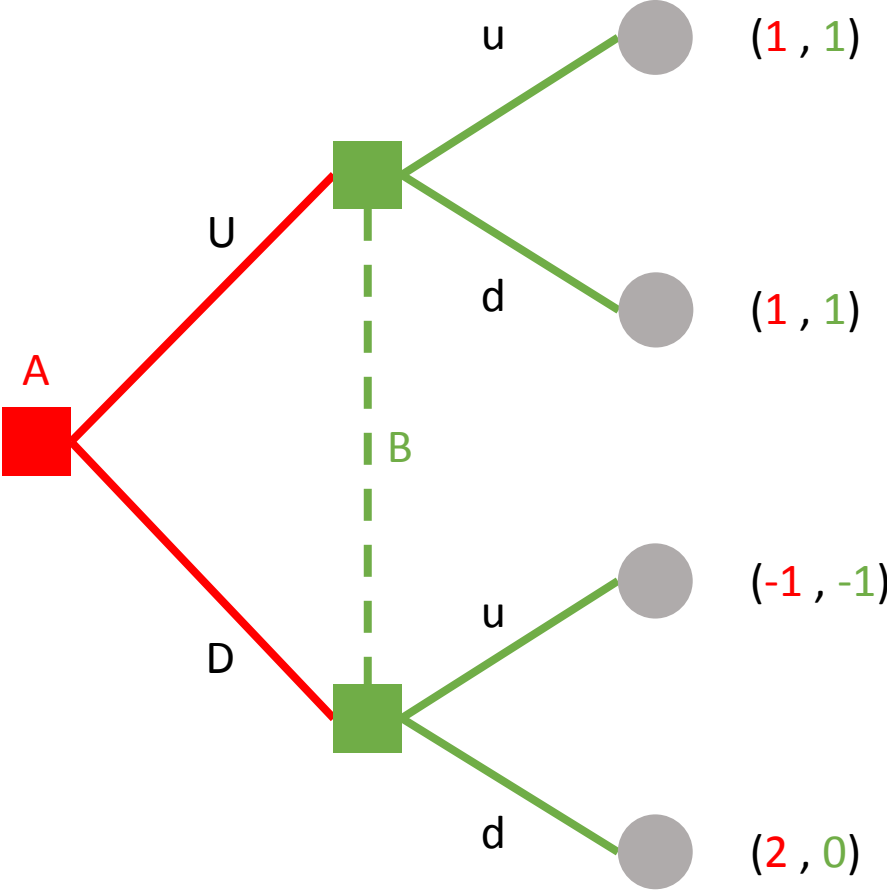
Selten's game from matrix
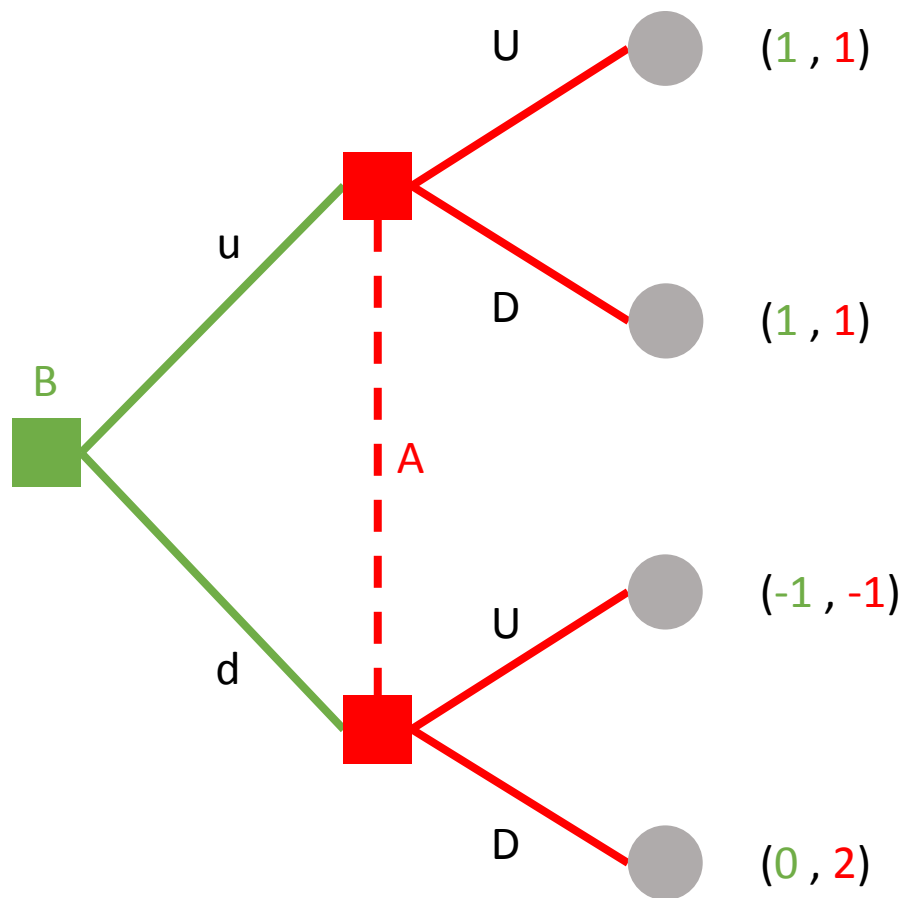
Selten's game – Nash equilibrium

B

	u	d
U	1 , 1	1 , 1
D	-1 , -1	2 , 0

A

Selten's game – Nash equilibrium

B

	u	d
U	**1** , 1	1 , 1
D	-1 , -1	**2** , 0

A

Selten's game – Nash equilibrium

B

	u	d
U	1 , 1	1 , 1
D	-1 , -1	2 , 0

A

Selten's game
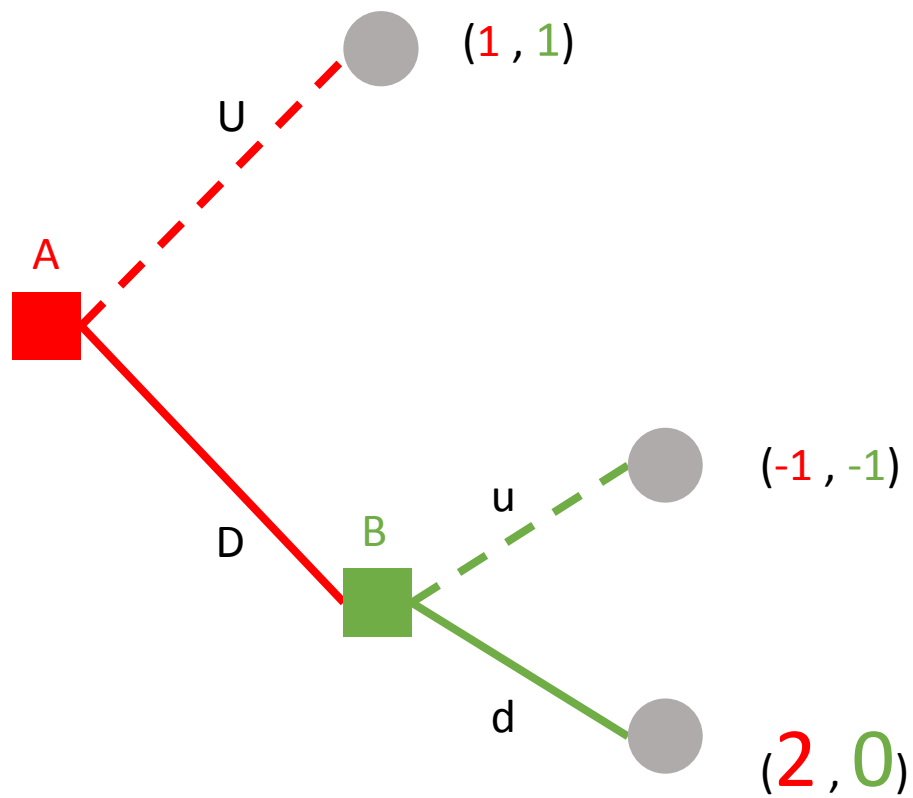
NE off the equilibrium path

- This game has another NE which is represented by action U which is not revealed in extensive form

- This equilibrium (U, u) is called a non-credible threat
  - B is willing to get its largest payoff, which will result from A playing U (deciding not to play the game)
  - B can change its payoff from A's decision about U and D only by threatening that it will play u if A plays D
  - But B will never play u, since it brings lower payoff than playing d

Selten's game and non-credible threat
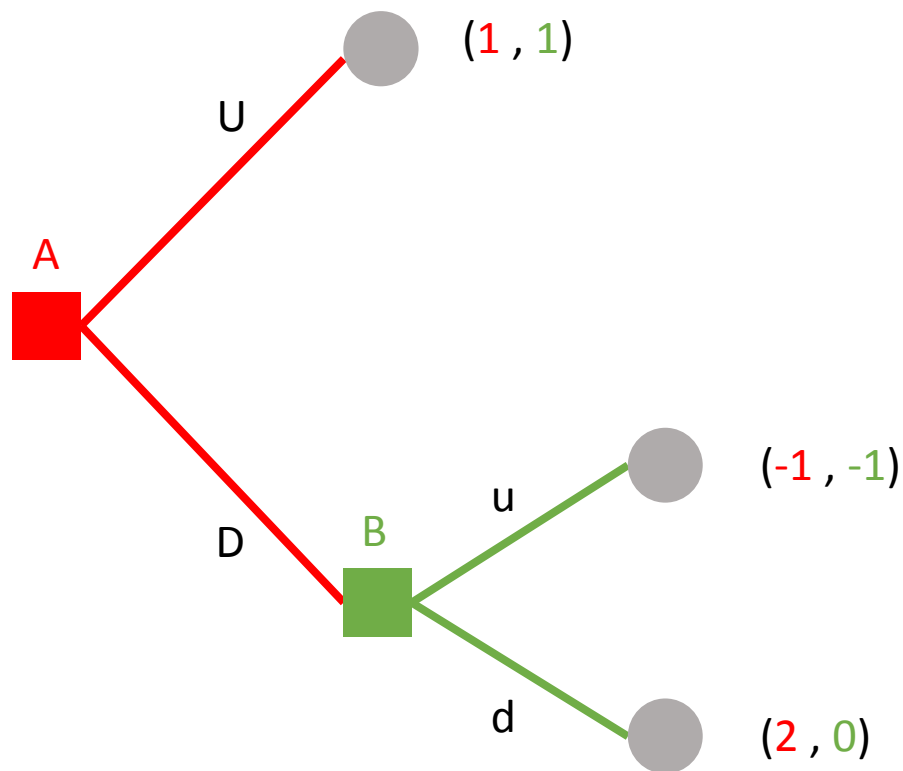
Subgame perfection

Subgame

- Subset of the nodes of a game starting with initial node and including all its successors that preserves all information sets of the game and over which a (new) game is defined by the restriction of the original game elements to these nodes

- Subgame can be treated as a game in its own right

- To ensure each player rational, there's requirement that strategies are optimal in all proper subgames

Subgames of Selten's game
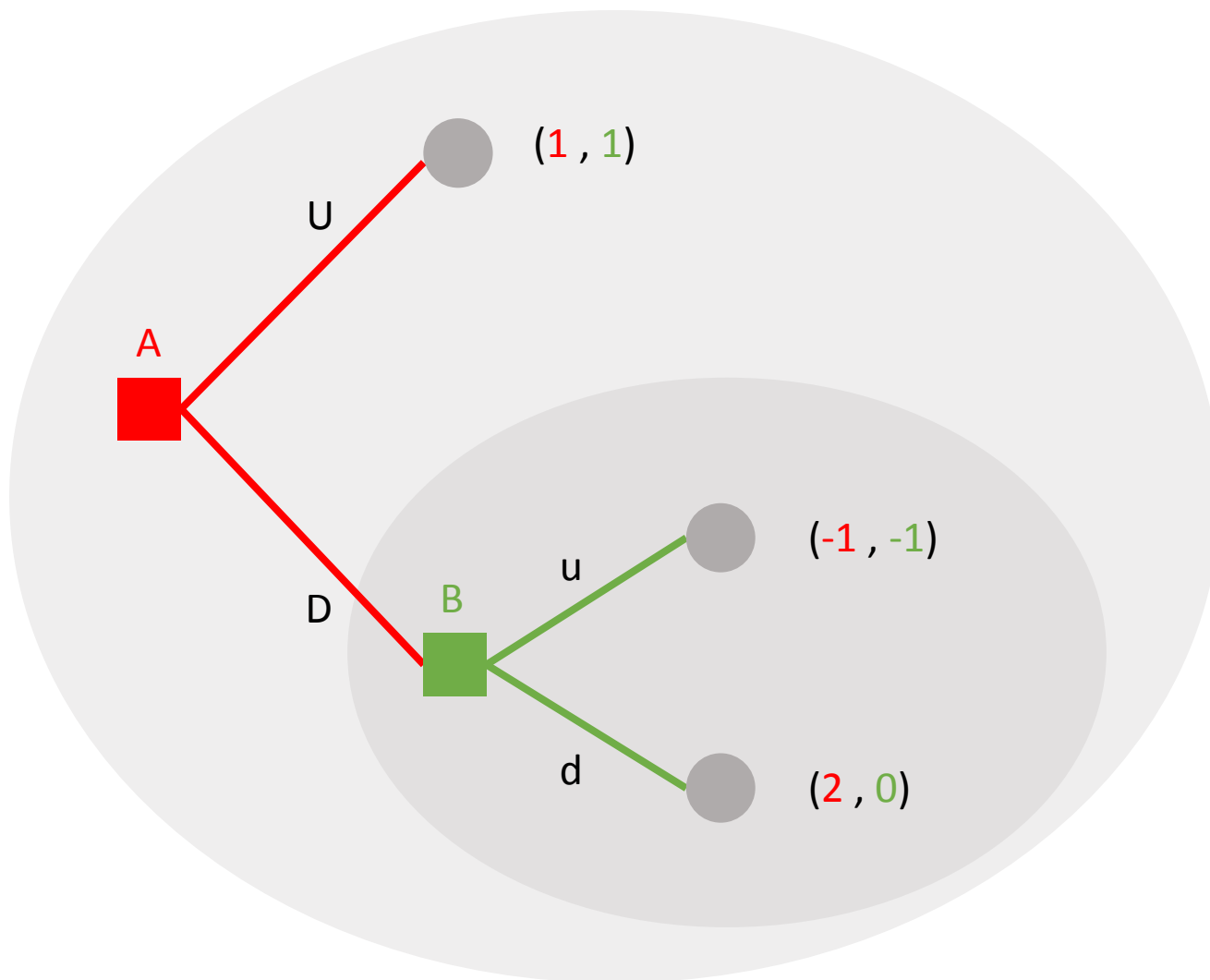
Information sets

- Player's information/knowledge about moves, which have been played in the game previously

- In games of perfect information, each node with its actions is information set – each player in each moment of the game knows, which moves were played before

- If player is uncertain about previous moves, information set contains more than one node

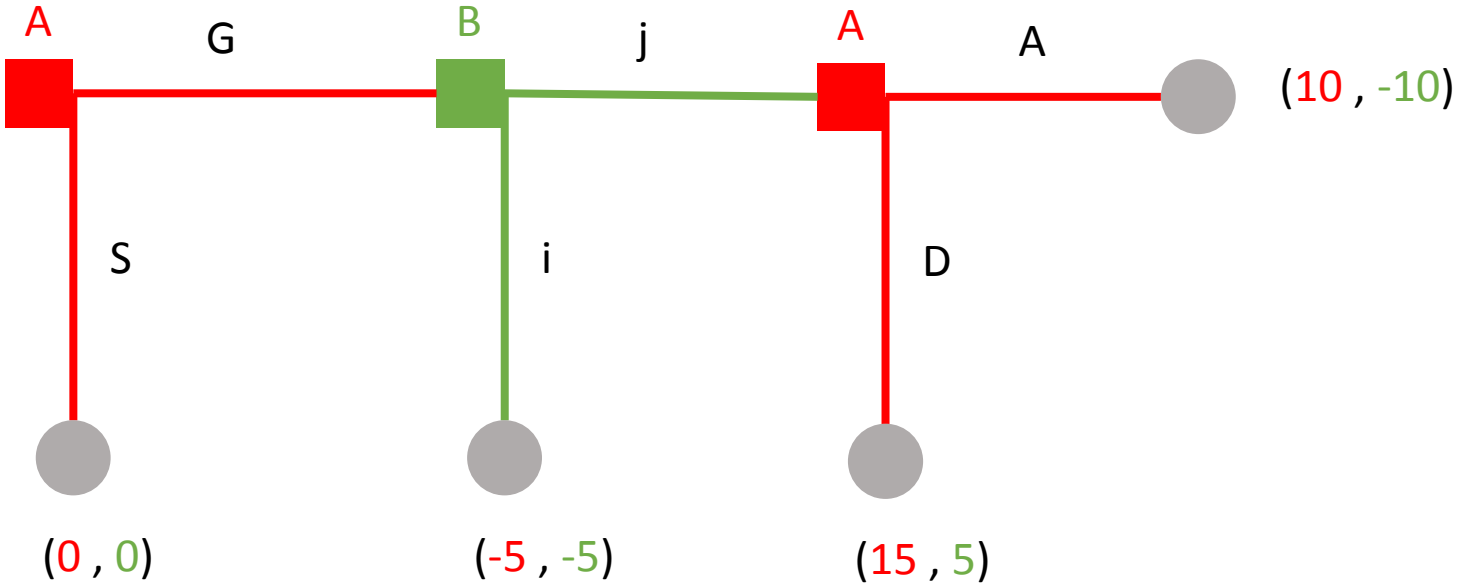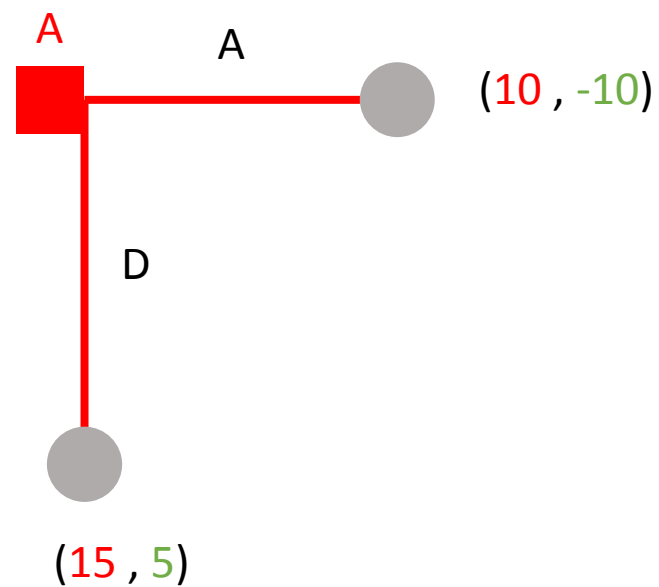- Information sets capture player's ignorance about previous moves in game
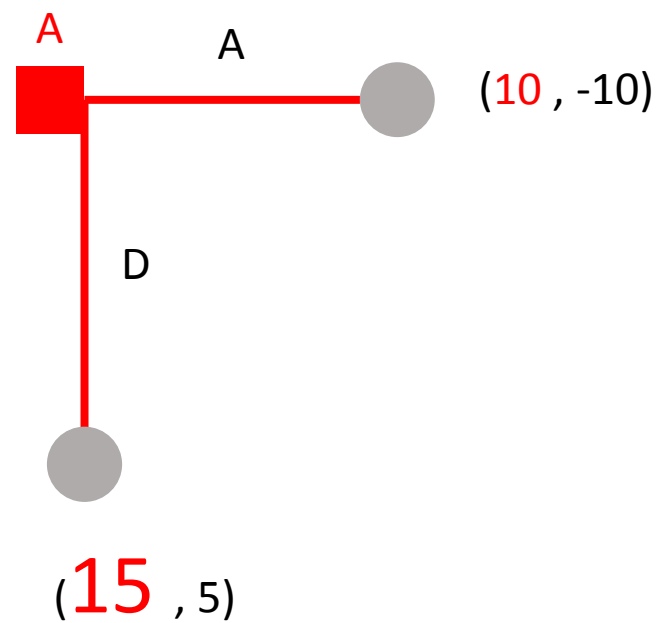
Subgames in PD in EF

A

C

D

B

c (5 , 5)

d (-10 , 10)

c (10 , -10)

d (-5 , -5)
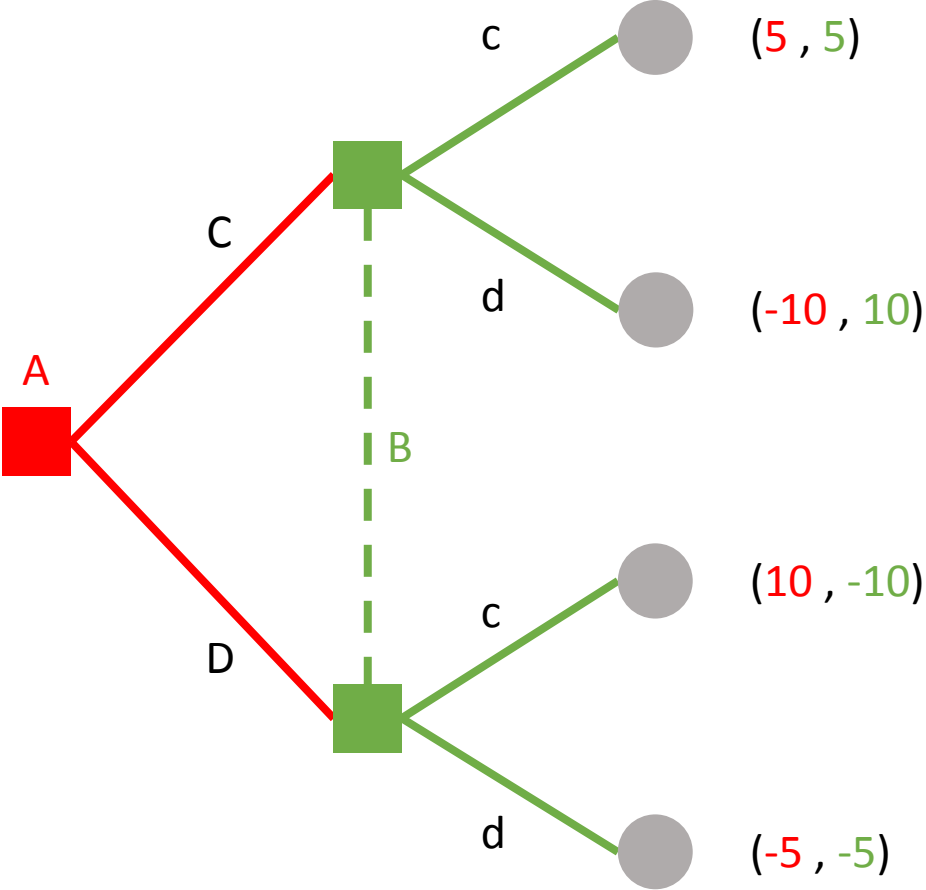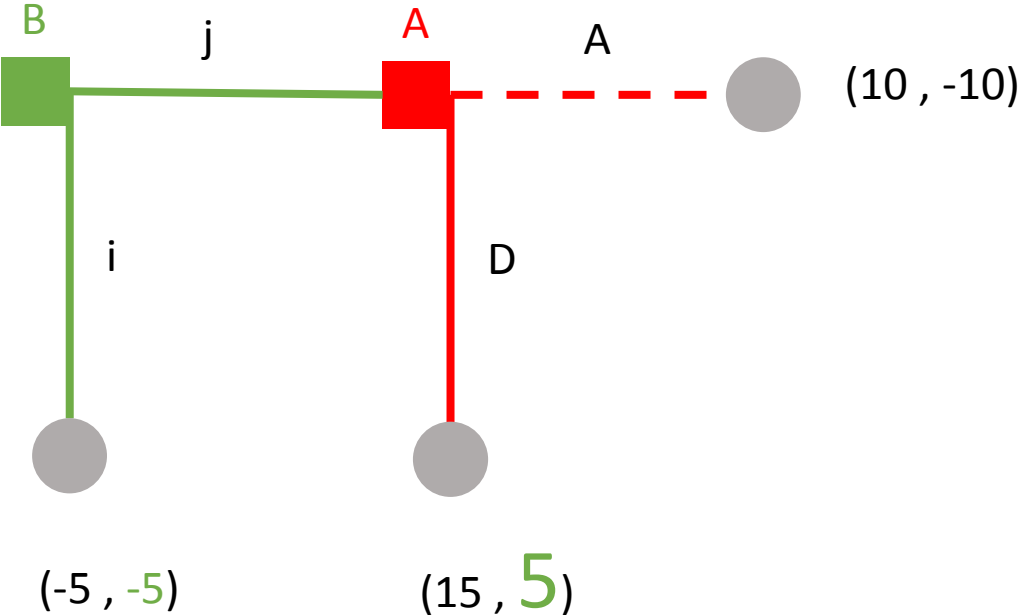
Subgames in PD in EF
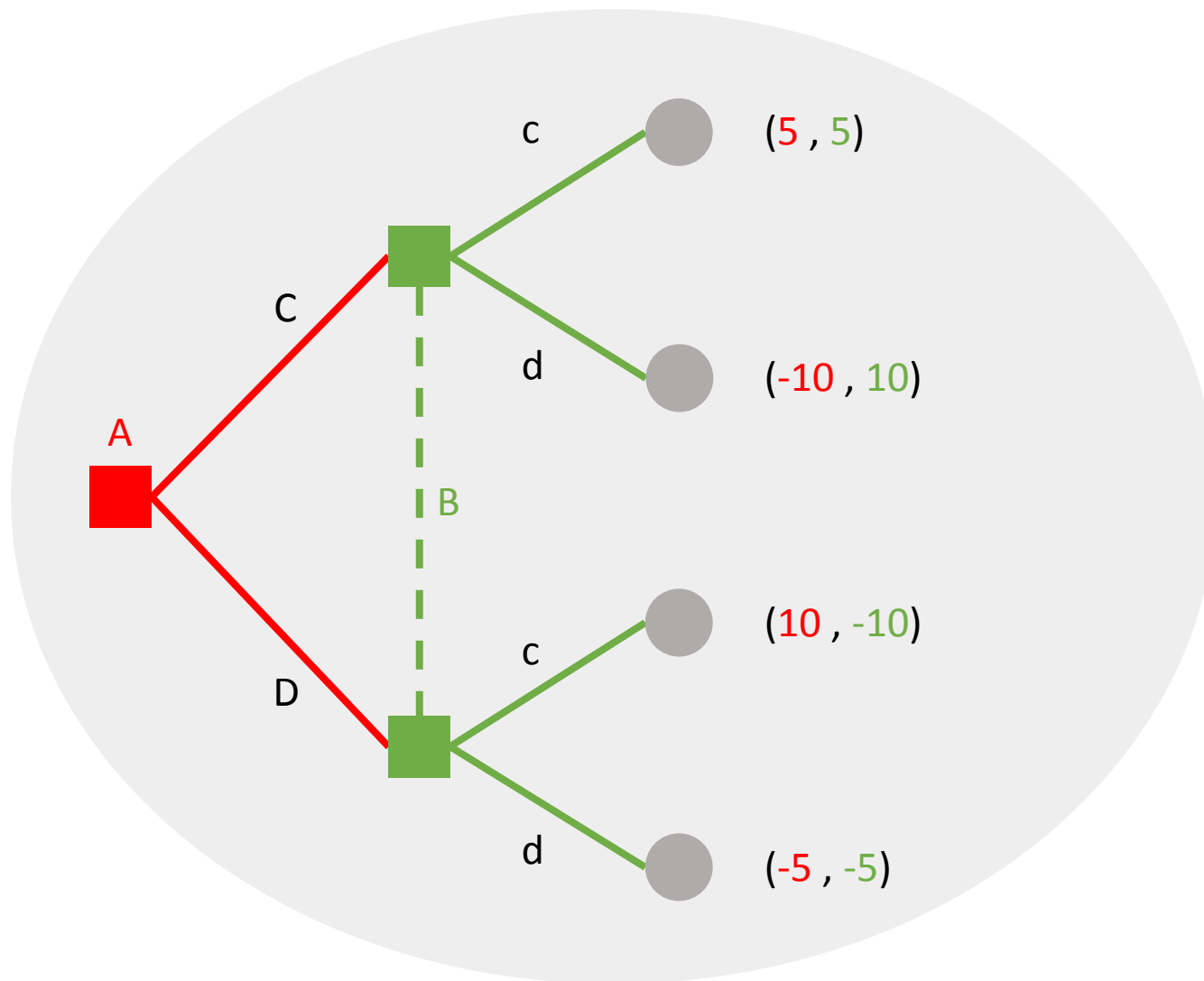
Game A
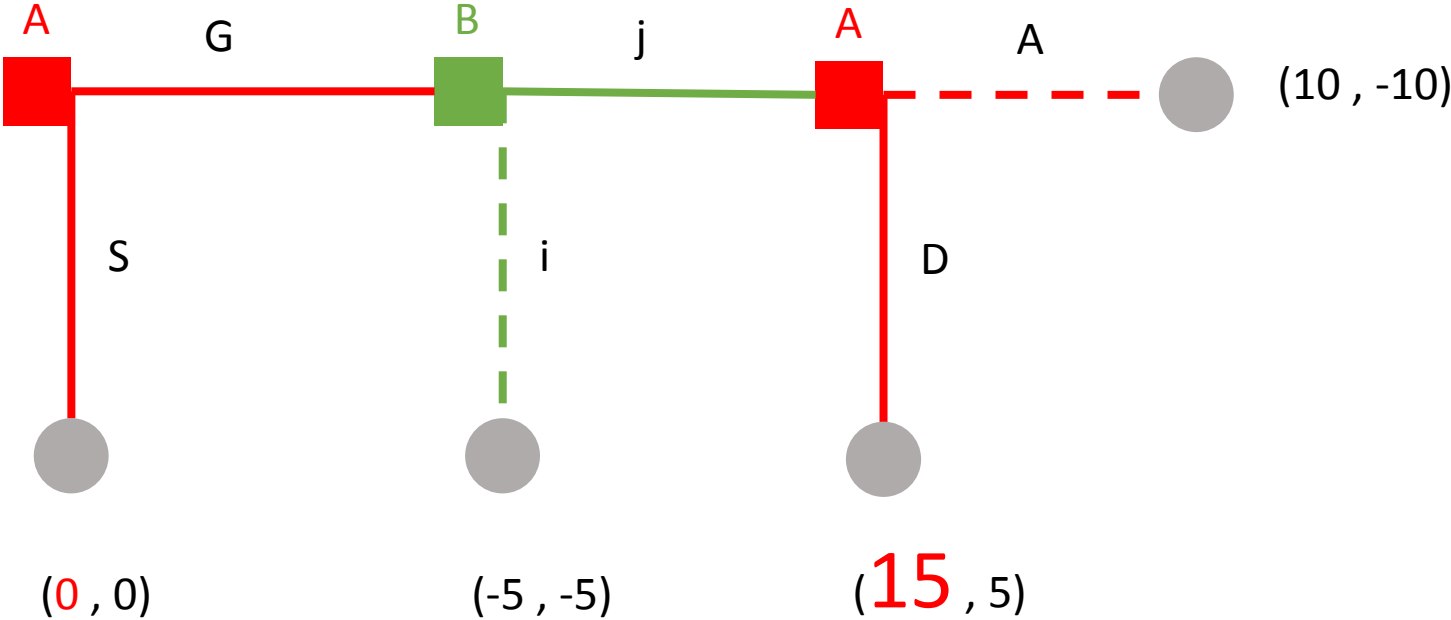
Game A – Backwards induction
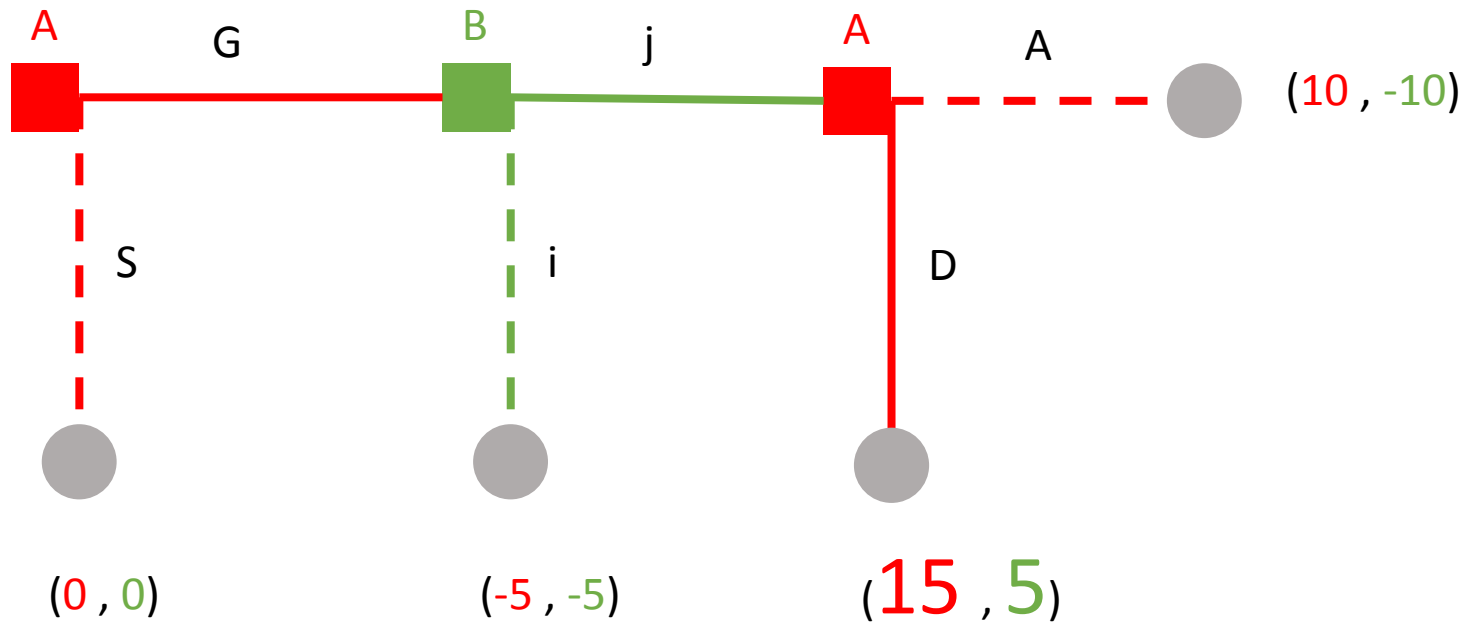
Game A – Backwards induction

Game A – Backwards induction

# Game A – Backwards induction

# Game A – Backwards induction solution

Subgame perfection

- Set of strategies is subgame perfect, if for every proper subgame, the restriction of those strategies to the subgame forms a Nash Equilibrium

- SPNE must thus form NE in every subgame of the game, from the game itself to the last subgame

- Restricts equilibriums in the game which are not a credible threat throughout the game

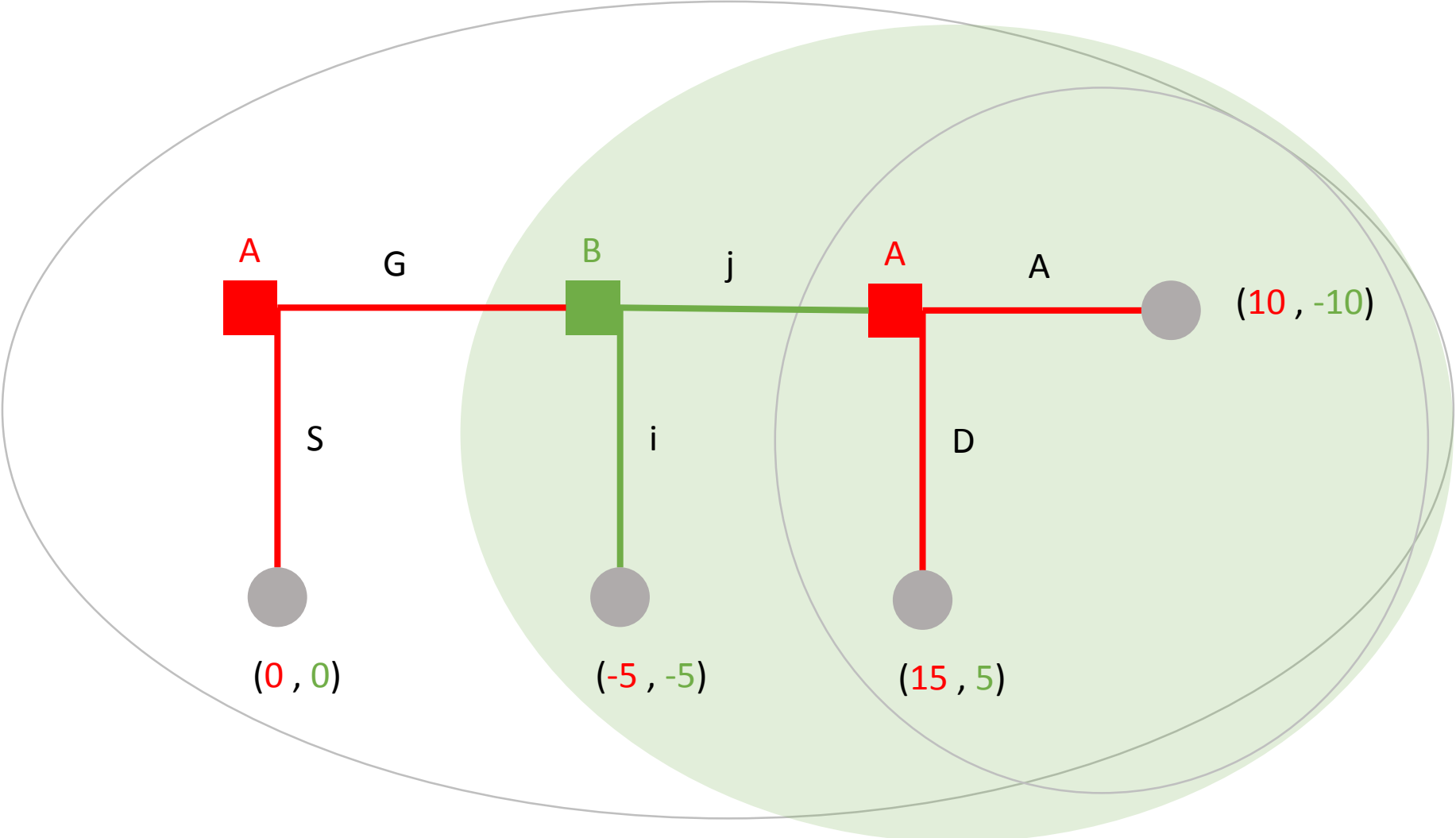# Game A - Subgames
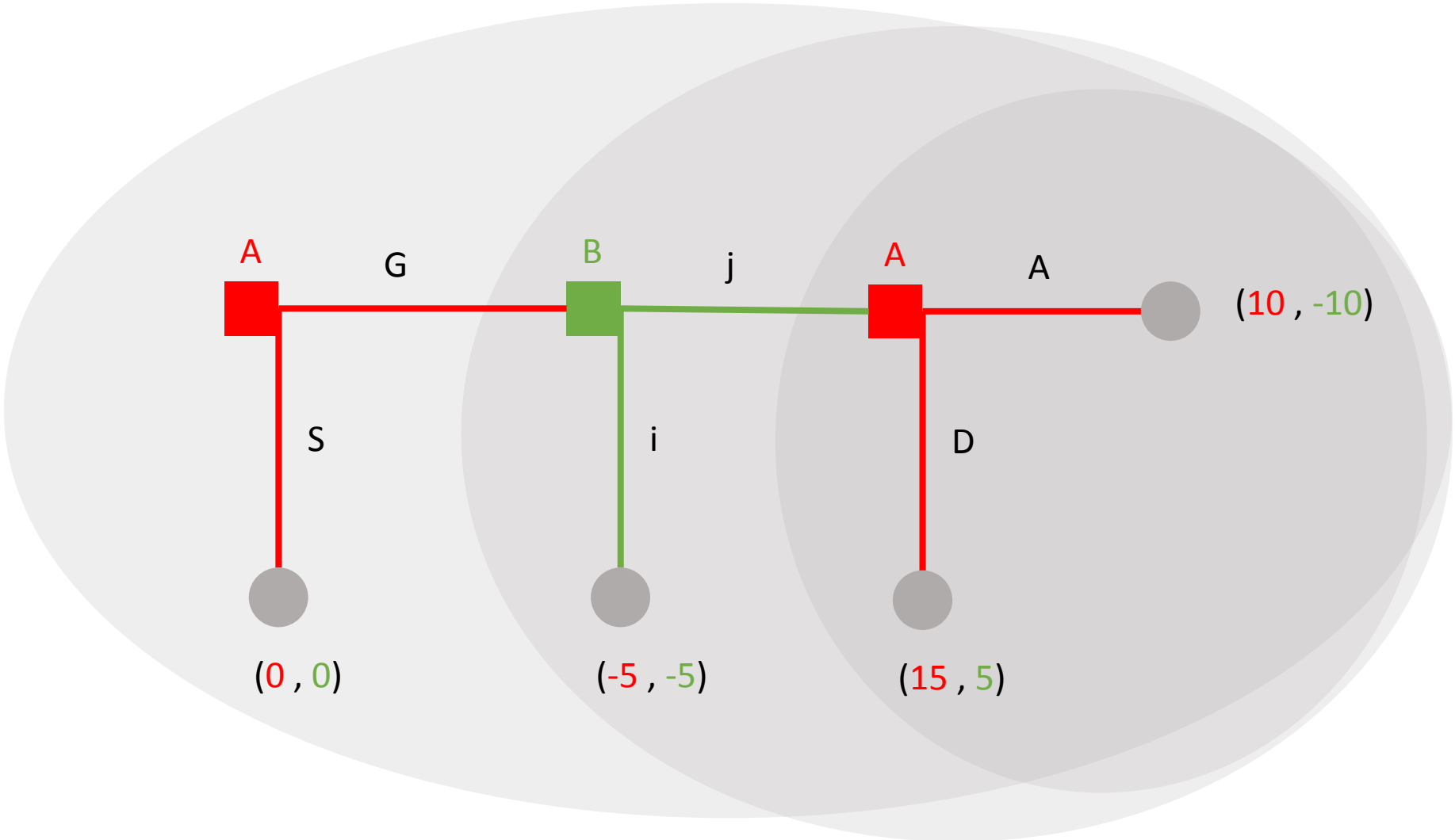
Game A – Subgame 1
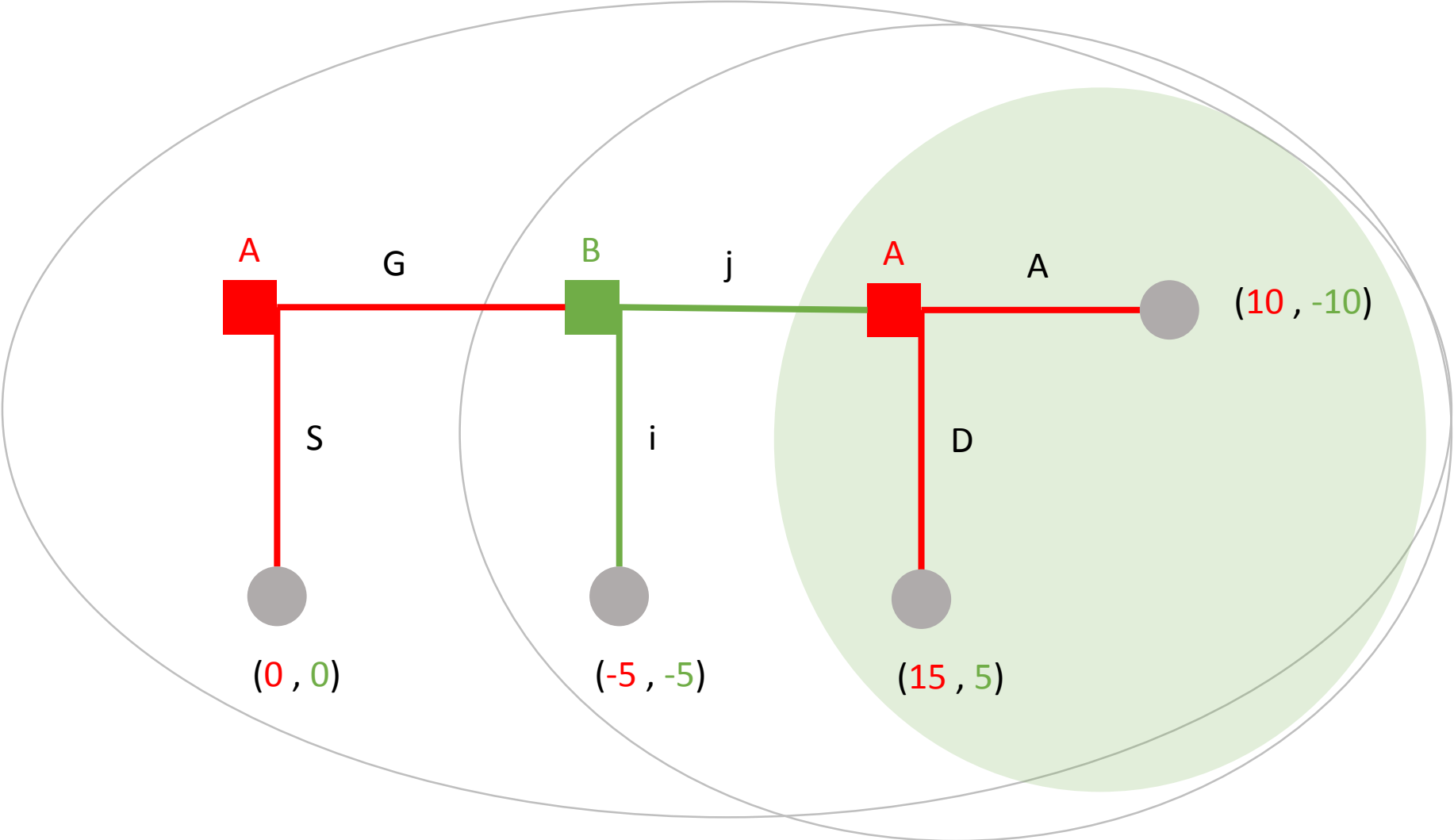
Strategic form of Subgame 1

B

	j
A	10 , -10
D	15 , 5

A

NE in strategic form of Subgame 1

B

j

	j
A	10 , -10
D	15 , 5

A

Game A – Subgame 2

Strategic form of Subgame 2

B

	j	i
A	10 , -10	-5 , -5
D	15 , 5	-5 , -5

A

NE in strategic form of Subgame 2

B

	J	i
A	10 , -10	-5 , -5
D	15 , 5	-5 , -5

A

Game A – Subgame 3

Strategic form of Subgame 3

B

	J	i
GA	10 , -10	-5 , -5
GD	15 , 5	-5 , -5
SA	0 , 0	0 , 0
SD	0 , 0	0 , 0

A
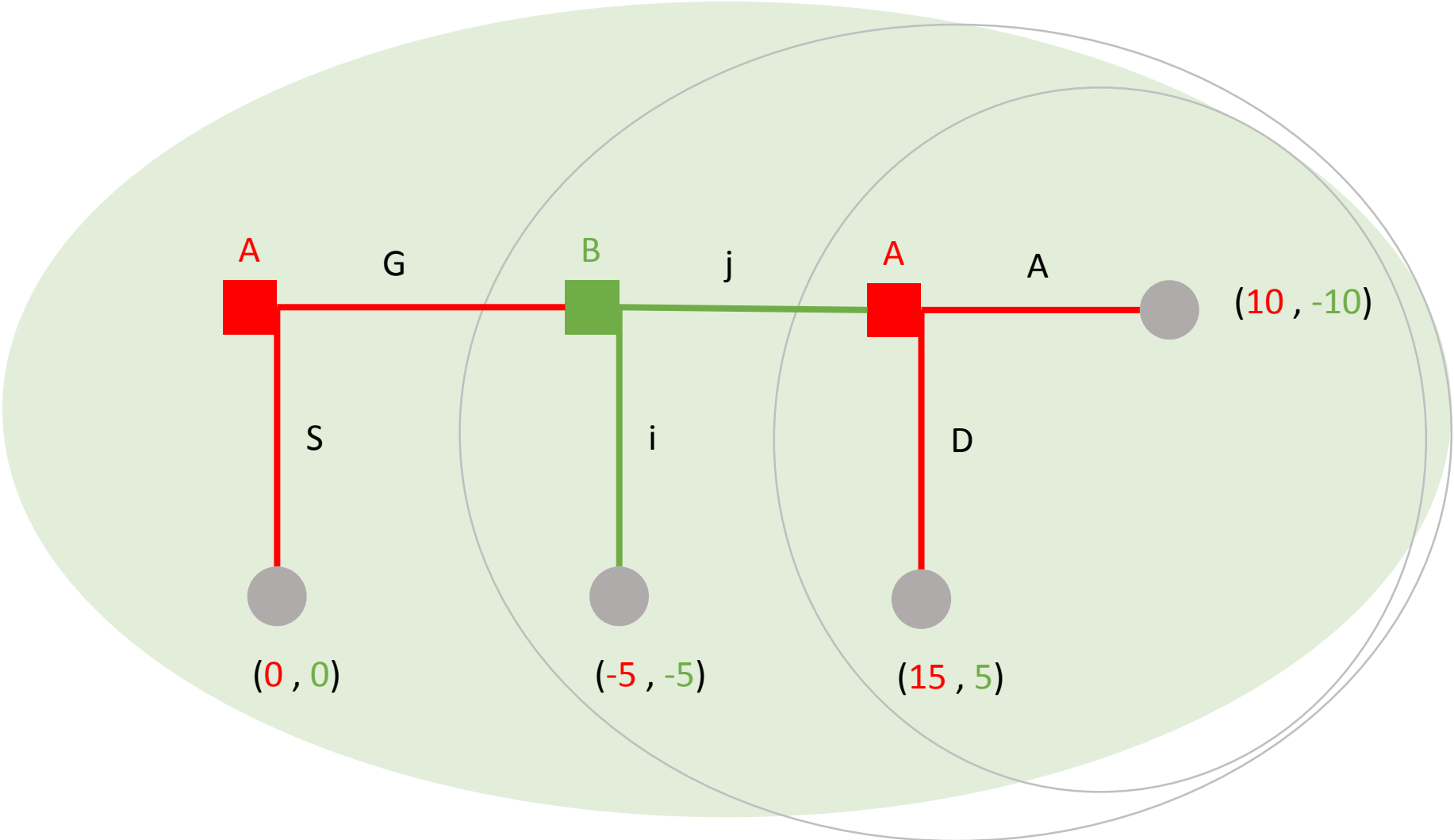
NE in strategic form of Subgame 3

B

		J	i
A	GA	10 , -10	-5 , **-5**
	GD	**15** , **5**	-5 , -5
	SA	0 , **0**	**0** , **0**
	SD	0 , **0**	**0** , **0**

NE in the Game A

- Strategic form NE
  - NE 1: <G, D; j>
  - NE 2: <S, A; i>
  - NE 3: <S, D; i>

- Backwards induction NE
  - NE 1: <G, D; j>

- Subgame perfect NE
  - NE 1: <G, D; j>

Game B

Strategic form of Subgame 1

B

u

L	0 , 2
R	-2 , -2

A
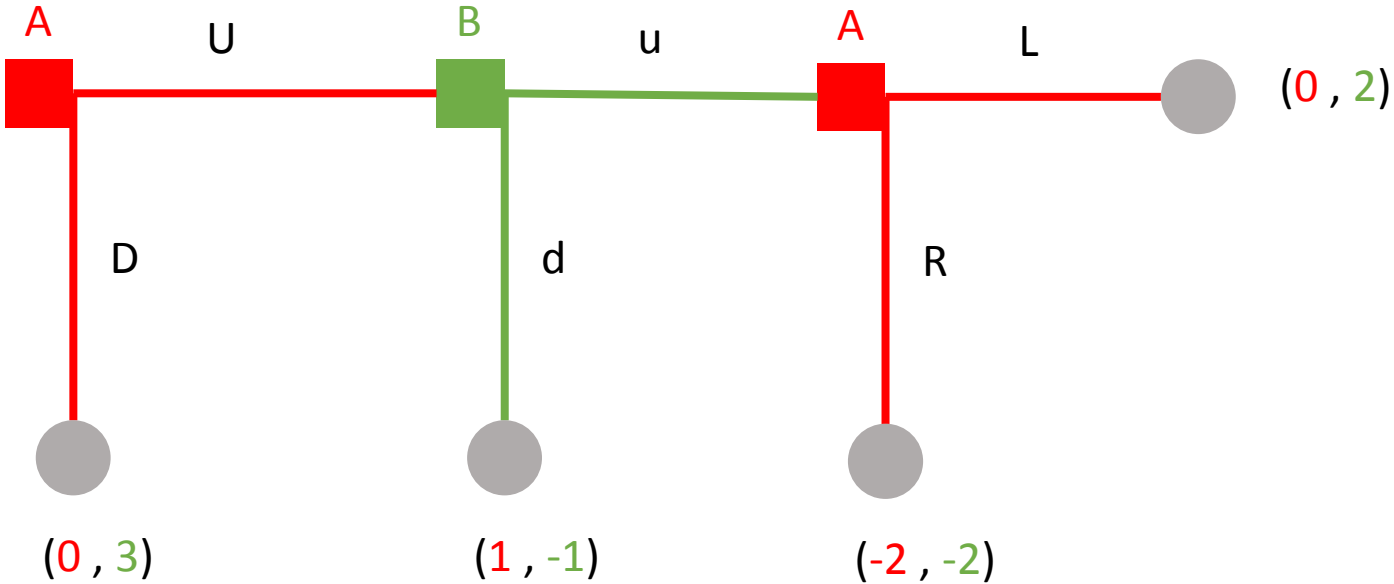
NE in strategic form of Subgame 1

B

	u
A L	**0** , 2
R	-2 , -2

Strategic form of Subgame 2

B

	u	d
L	0 , 2	1 , -1
R	-2 , -2	1 , -1

A

NE in strategic form of Subgame 2

	u	d
L	0 , 2	1 , -1
R	-2 , -2	1 , -1

B

A

Strategic form of Subgame 3

B

	u	d
UL	0 , 3	0 , 3
UR	0 , 3	0 , 3
DL	0 , 2	1 , -1
DR	-2 , -2	1 , -1

A

NE in strategic form of Subgame 3

B

	u	d
UL	0 , 3	0 , 3
UR	0 , 3	0 , 3
DL	0 , 2	1 , -1
DR	-2 , -2	1 , -1

A

NE in the game

- Strategic form NE
  - NE 1: <U, L; u>
  - NE 2: <U, R; u>
  - NE 3: <D, L; u>
  - NE 4: <D, R; d>

- Backwards induction NE
  - NE 2: <U, R; u>
  - NE 3: <D, L; u>

- Subgame perfect NE
  - NE 3: <D, L; u>
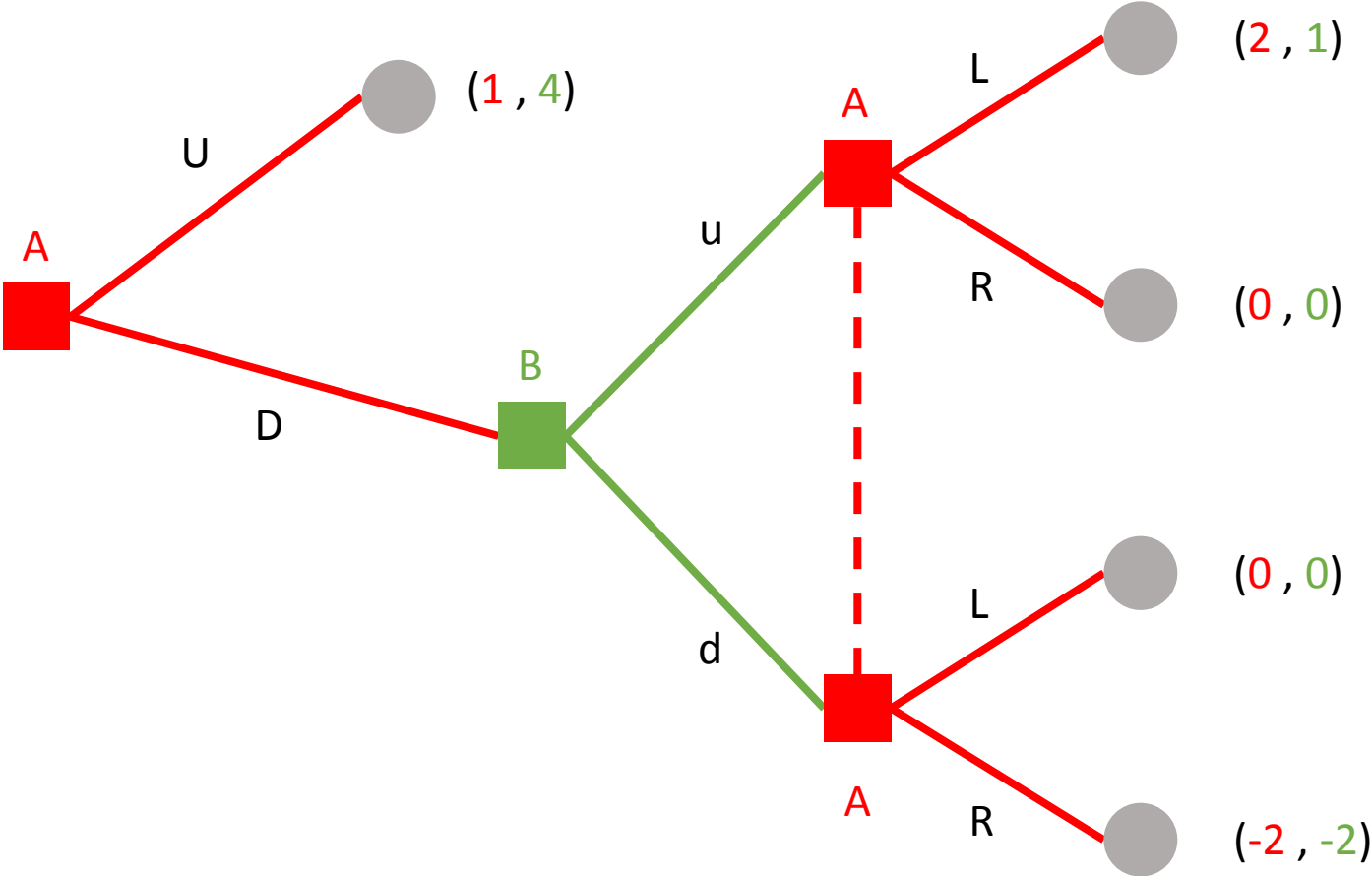
Game X

Game X - Subgames

Strategic form of Subgame of Game X

B

	u	d
L	2 , 1	0 , 0
R	0 , 0	-2 , -2

A
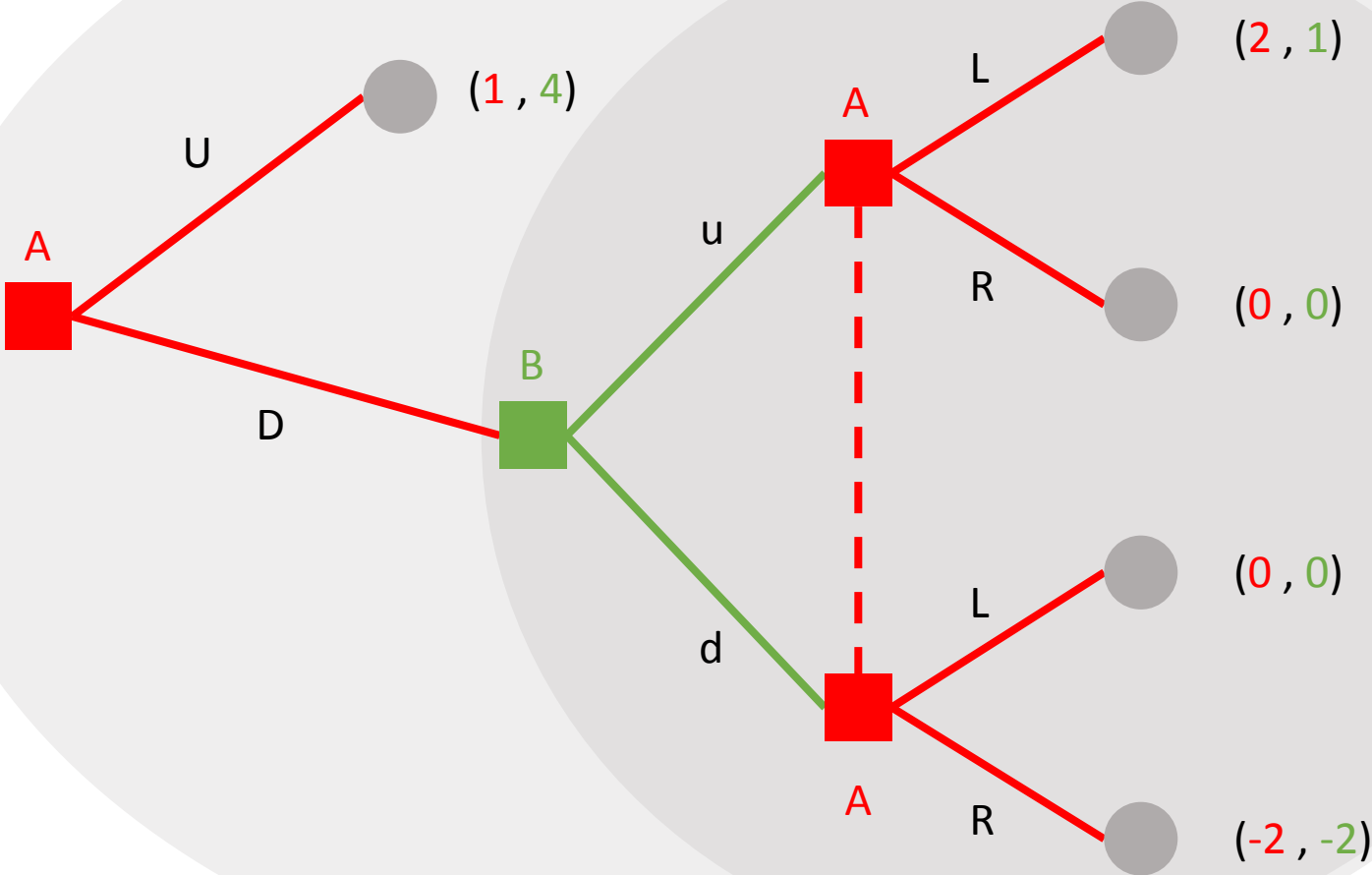
NE in strategic form of Subgame of Game X

	B	
	u	d
A L	**2** , **1**	**0** , 0
R	0 , **0**	-2 , -2

Strategic form of the whole Game X

B

	u	d
UL	1 , 4	1 , 4
UR	1 , 4	1 , 4
DL	2 , 1	0 , 0
DR	0 , 0	-2 , -2

A

NE in strategic form of the whole Game X

NE in the Game X

- Strategic form NE
  - NE 1: <U, L; d>
  - NE 2: <U, R; d>
  - NE 3: <D, L; u>

- Backwards induction NE
  - Can't apply

- Subgame perfect NE
  - NE 3: <D, L; u>

Thank you for your attention

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