Game theory 2

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Sum-up of the previous lecture

		Opponent s_1	Opponent s_3
Me	S_2	0 , 3	6 , 4
Me	S_3	1 , 5	5 , 2

Social welfare

Social welfare

- Situation where **sum of all payoffs** of an outcome is at its **maximum**
- Might lead to rationally unstable solutions
- Does not provide a solid analytical tool

Game M

		B	
		Right	Left
A	Right	2 , 2	0 , 0
	Left	0 , 0	1 , 1

Game M – Social welfare

B

		Right	Left
A	Right	4	0
	Left	0	2

Game N

B

A		l	r
	L	2 , 2	4 , 0
	R	2 , 3	8 , -1

Game N – Social welfare

B

A		l	r
	L	4	4
	R	5	7

Prisoner's dilemma – Social welfare

A \ B	c	d
C	5 , 5	0 , 7
D	7 , 0	3 , 3

Prisoner's dilemma – Social welfare

		B	
		c	d
A	C	10	7
	D	7	6

Prisoner's dilemma – Social welfare

B

A		c	d
	C	5 , 50	0 , 70
	D	7 , 0	3 , 30

Prisoner's dilemma – Social welfare

B

A	c	d
C	55	70
D	70	33

Prisoner’s dilemma – Social welfare

B

A	c	d
C	50 , 5	0 , 7
D	70 , 0	30 , 3

Prisoner's dilemma – Social welfare

A \ B	C	D
C	55	7
D	70	33

Pareto efficiency

Game M

		B: Right	B: Left
A	Right	2 , 2	0 , 0
	Left	0 , 0	1 , 1

Game M

		B: Right	B: Left
A	Right	2 , 2	0 , 0
A	Left	0 , 0	1 , 1

Game M – pure strategy equilibriums

		B	
		Right	Left
A	Right	2 , 2	0 , 0
	Left	0 , 0	1 , 1

Pareto efficiency

- Outcome is Pareto efficient (Pareto optimal), if there is **no other outcome** which is **better or equal for all players** and **strictly better for some player**

- Conversely, outcome A is **Pareto dominated**, if there is outcome B that makes **all players as good** (weakly better) and **one player strictly better** compared to outcome A

- **Pareto dominated** outcome is **not Pareto efficient**

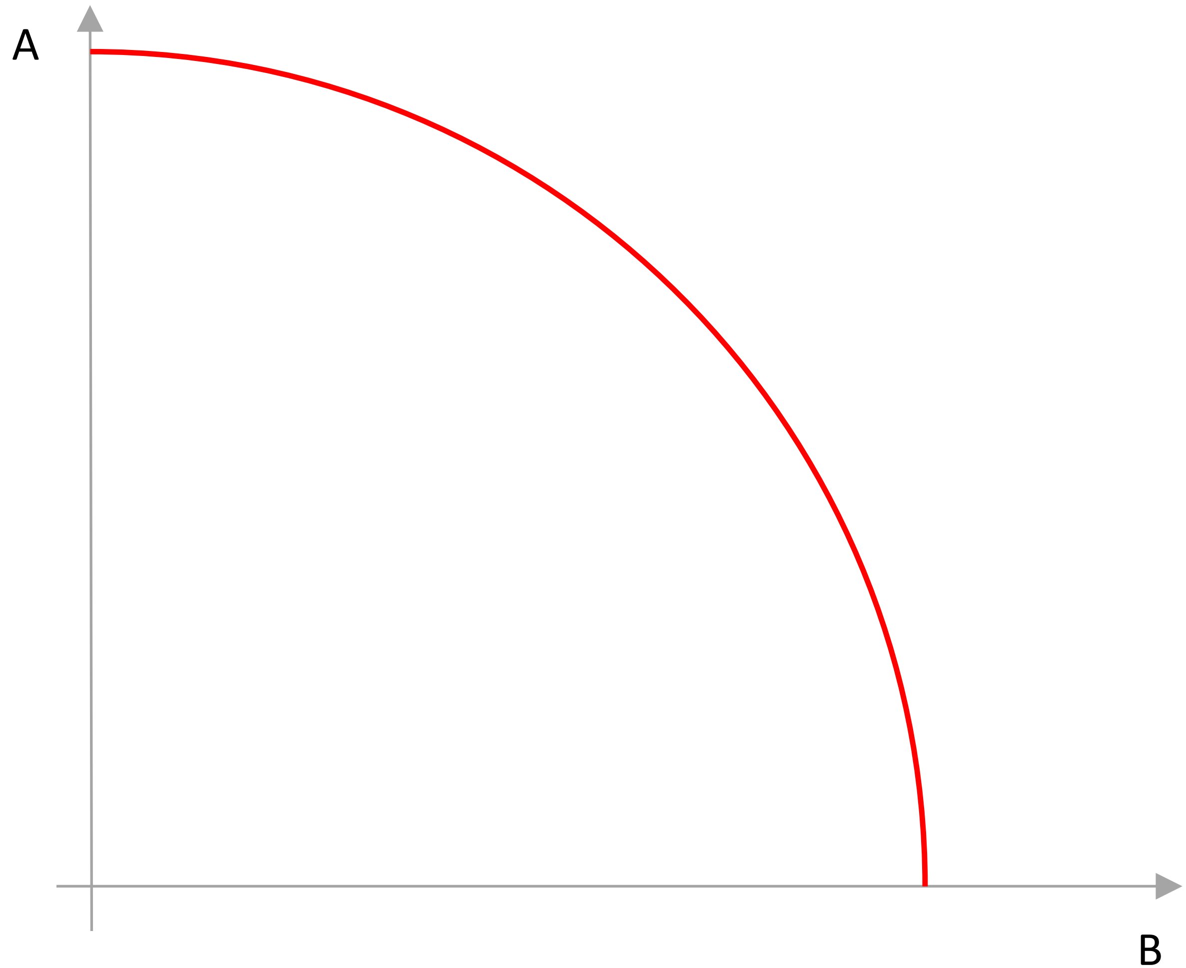
- Might lead to rationally unstable solutions

Game M – Pareto efficiency

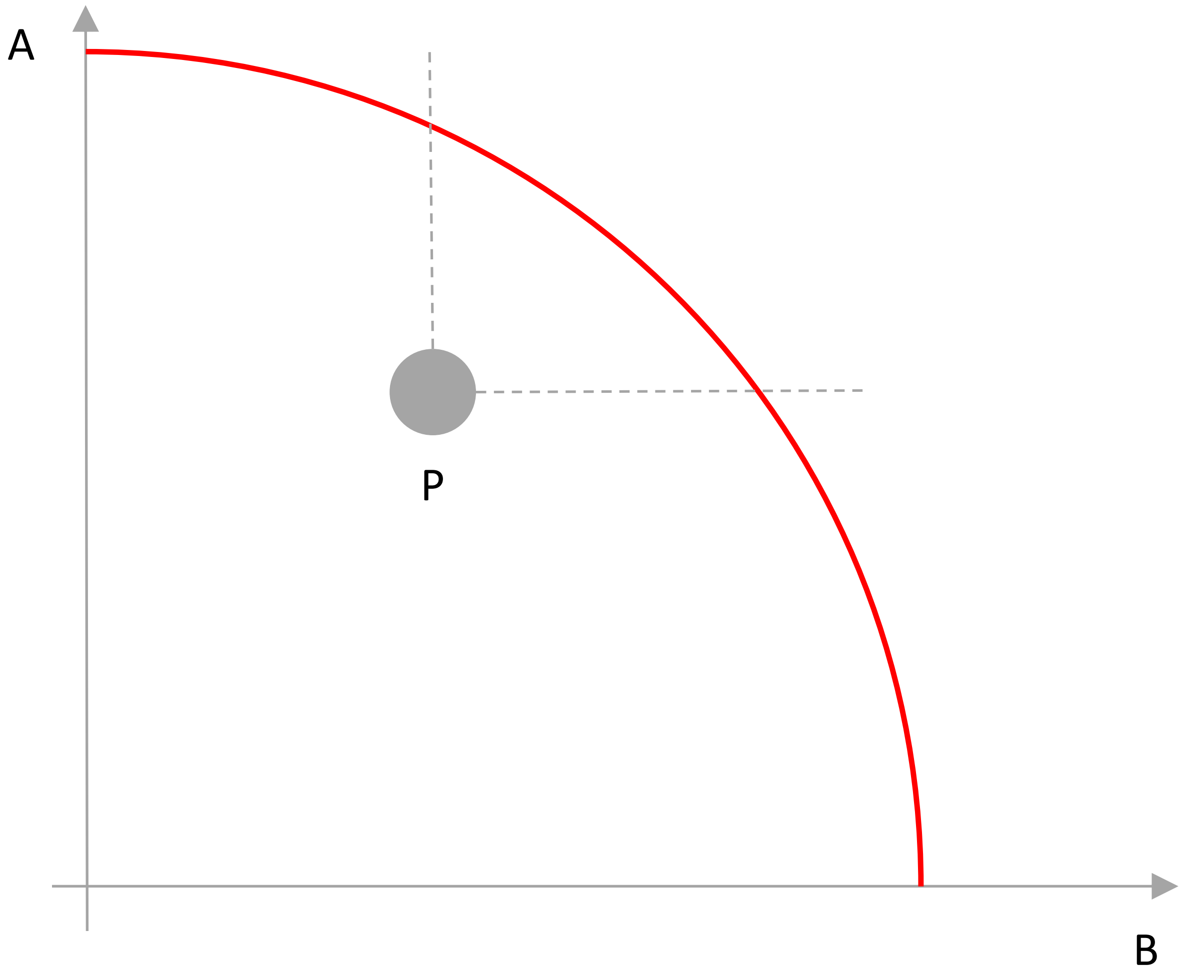
B

A	Right	Left
Right	2 , 2	0 , 0
Left	0 , 0	1 , 1

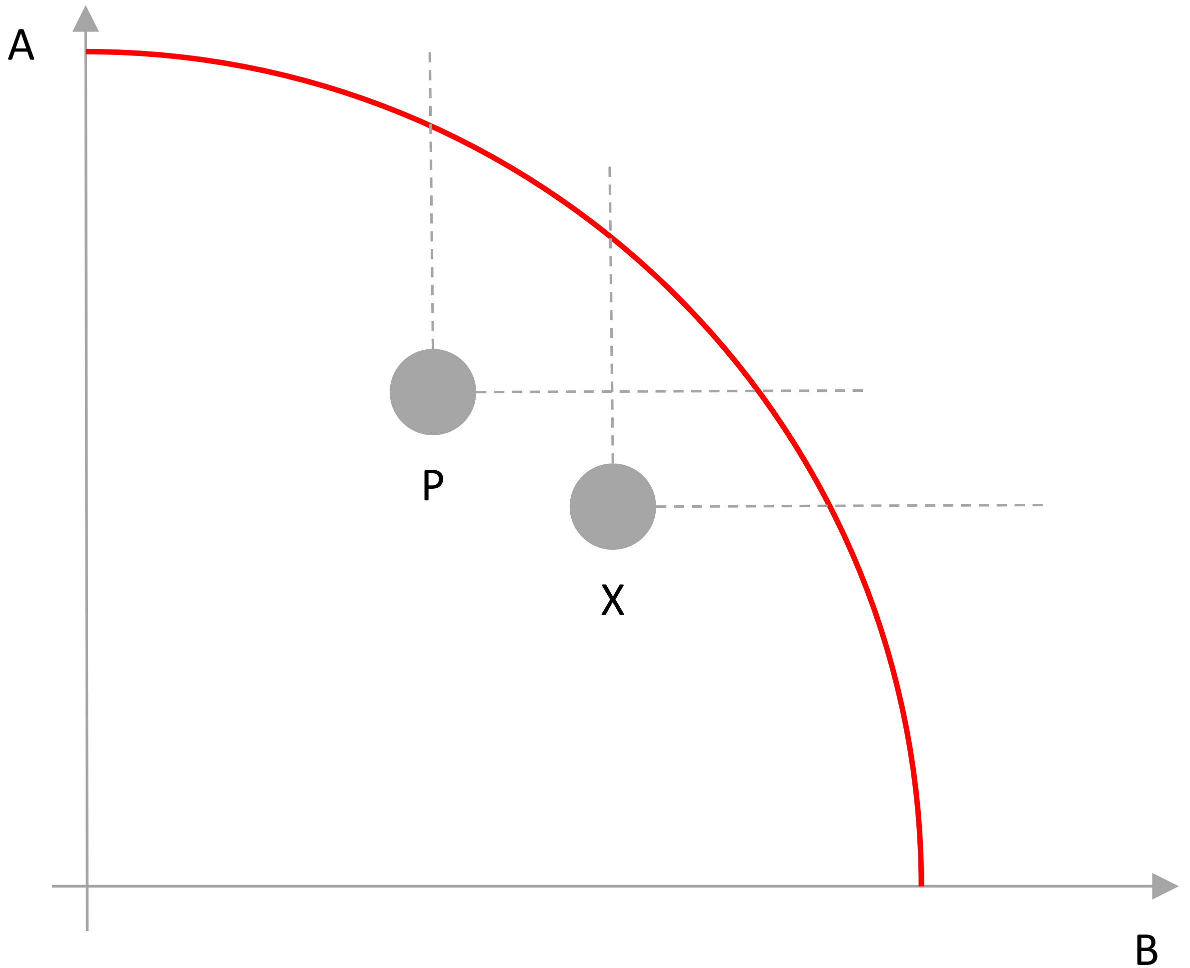
Pareto efficiency



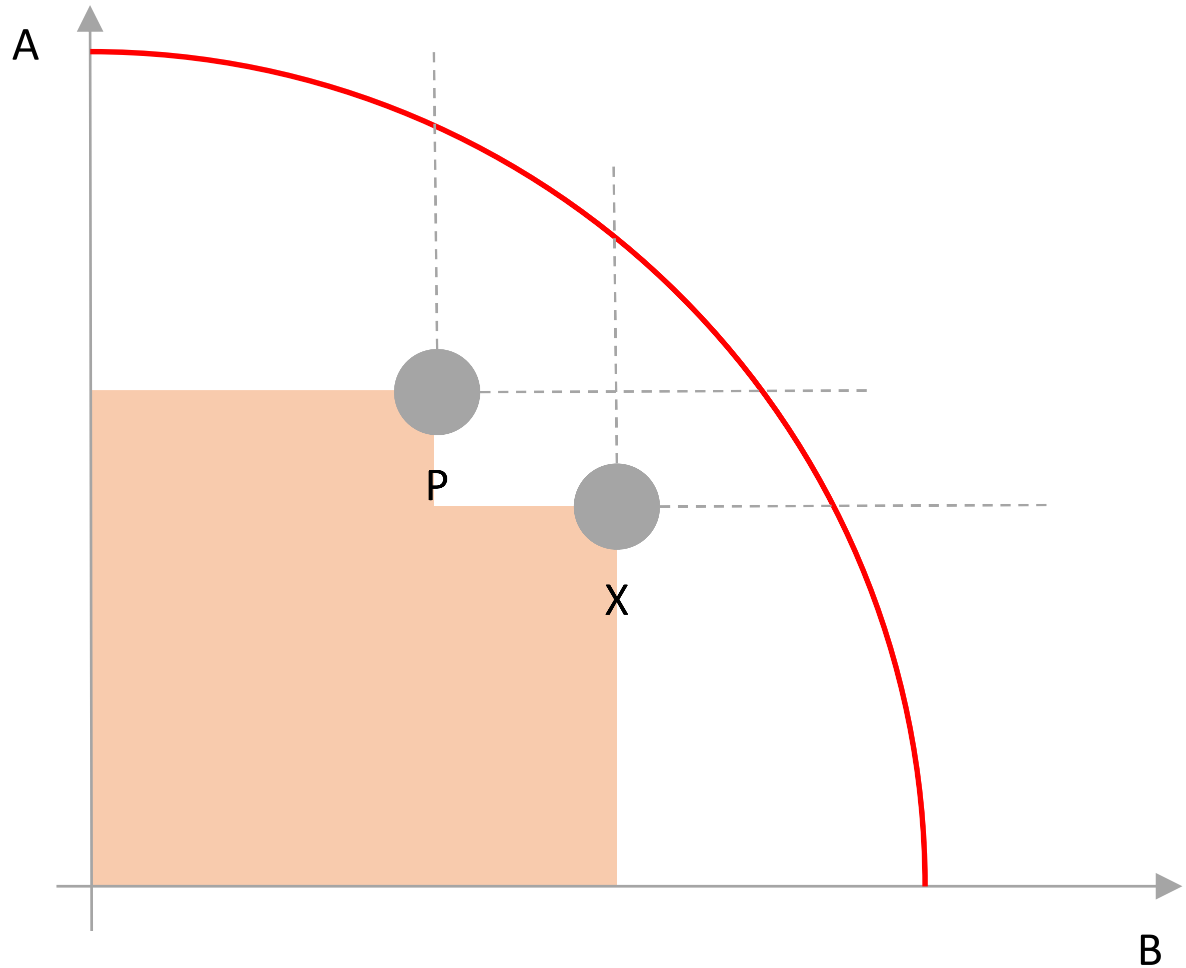
Pareto efficiency



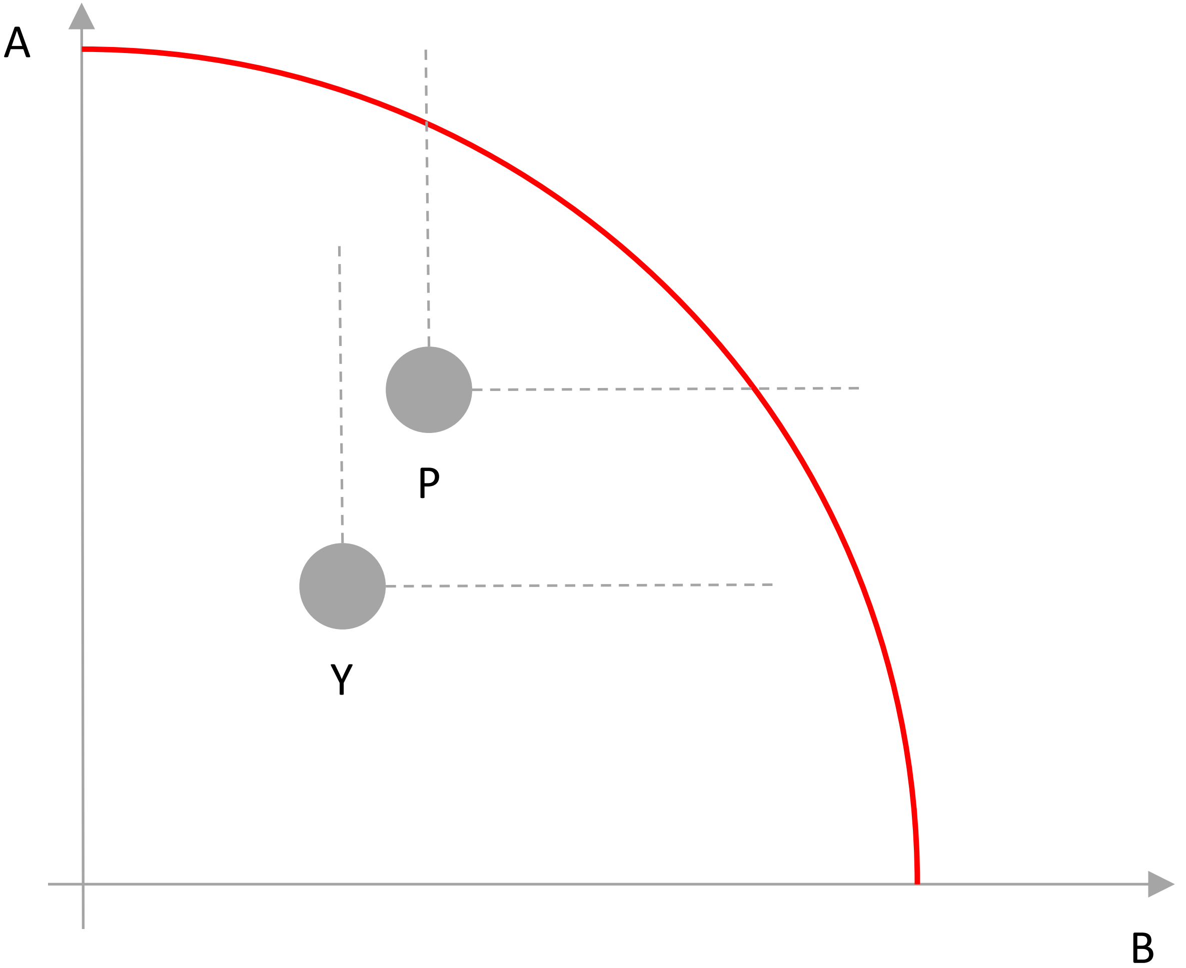
Pareto efficiency



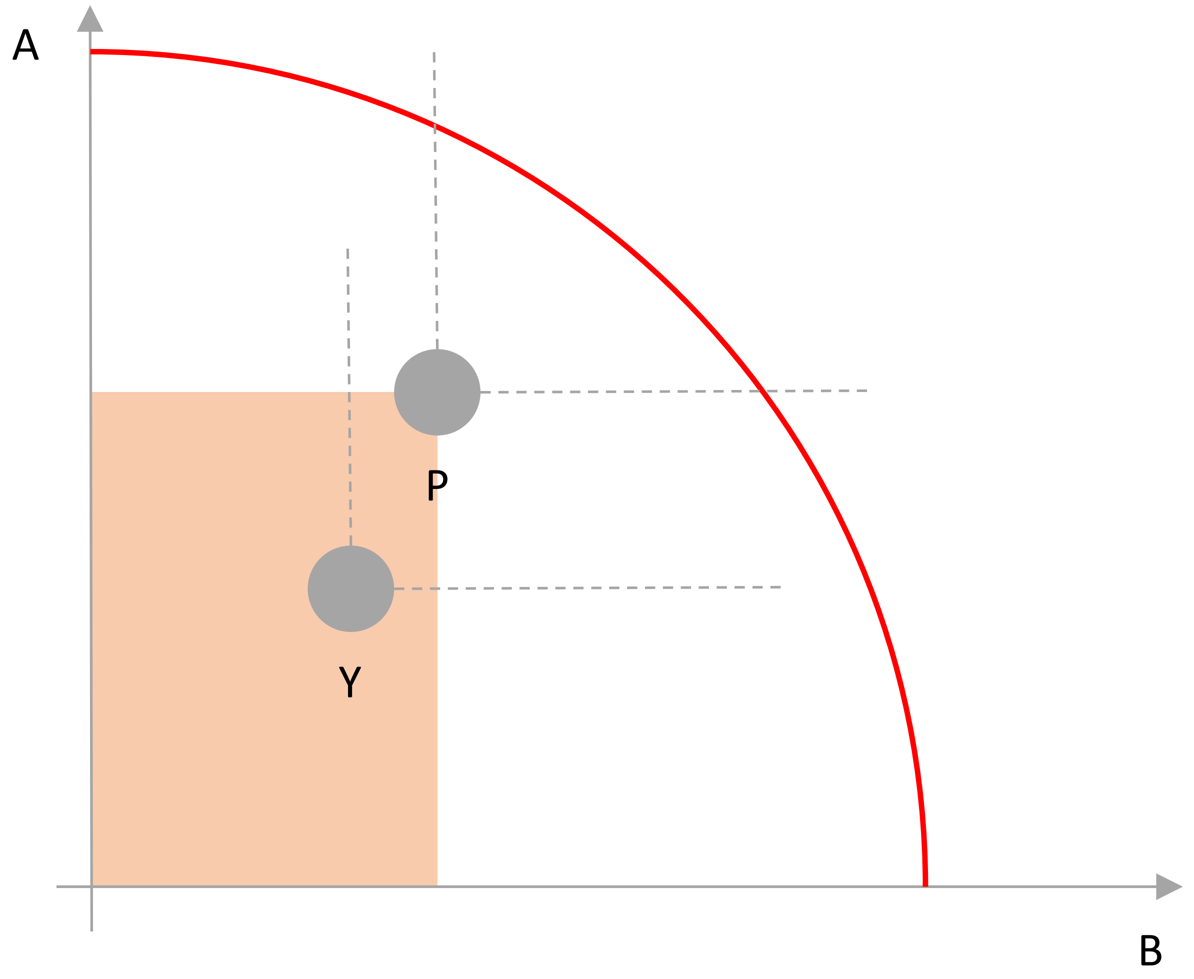
Pareto efficiency



Pareto efficiency



Pareto efficiency



Prisoner’s dilemma – Pareto efficiency

B

A	c	d
C	5 , 5	0 , 7
D	7 , 0	3 , 3

Prisoner's dilemma – Pareto efficiency

B

A		c	d
	C	5 , 5	0 , 7
	D	7 , 0	3 , 3

Prisoner's dilemma – Pareto efficiency

		B	
		c	d
A	C	5 , 5	0 , 7
	D	7 , 0	3 , 3

Game N

		B: l	B: r
A	L	2 , 2	4 , 0
A	R	2 , 3	8 , -1

Game N – Pareto efficiency

		B	
		l	r
A	L	2 , 2	4 , 0
	R	2 , 3	8 , -1

Game N – Pareto efficiency

		B	
		l	r
A	L	2 , 2	4 , 0
	R	2 , 3	8 , -1

Pareto optimality solid tool for comparing equilibriums

Mixed-strategy Nash equilibrium

Matching pennies

- Two players
- Players choose heads or tails
- If players match heads/tails, I (Player 1) win both coins
- If players don’t match heads/tails, opponent (Player 2) wins both coins

Matching pennies

My pair

Me

	Heads	Tails
Heads	1 , -1	-1 , 1
Tails	-1 , 1	1 , -1

Matching pennies – Pareto efficiency

My pair

Me

	Heads	Tails
Heads	1 , -1	-1 , 1
Tails	-1 , 1	1 , -1

Matching pennies – mixed strategy

My pair

Me		Heads (0.5)	Tails (0.5)
	Heads (0.5)	1 , -1	-1 , 1
	Tails (0.5)	-1 , 1	1 , -1

Calculation
of mixed-strategy NE

Game Y

B

A		L	R
	U	3 , -3	-2 , 2
	D	-1 , 1	0 , 0

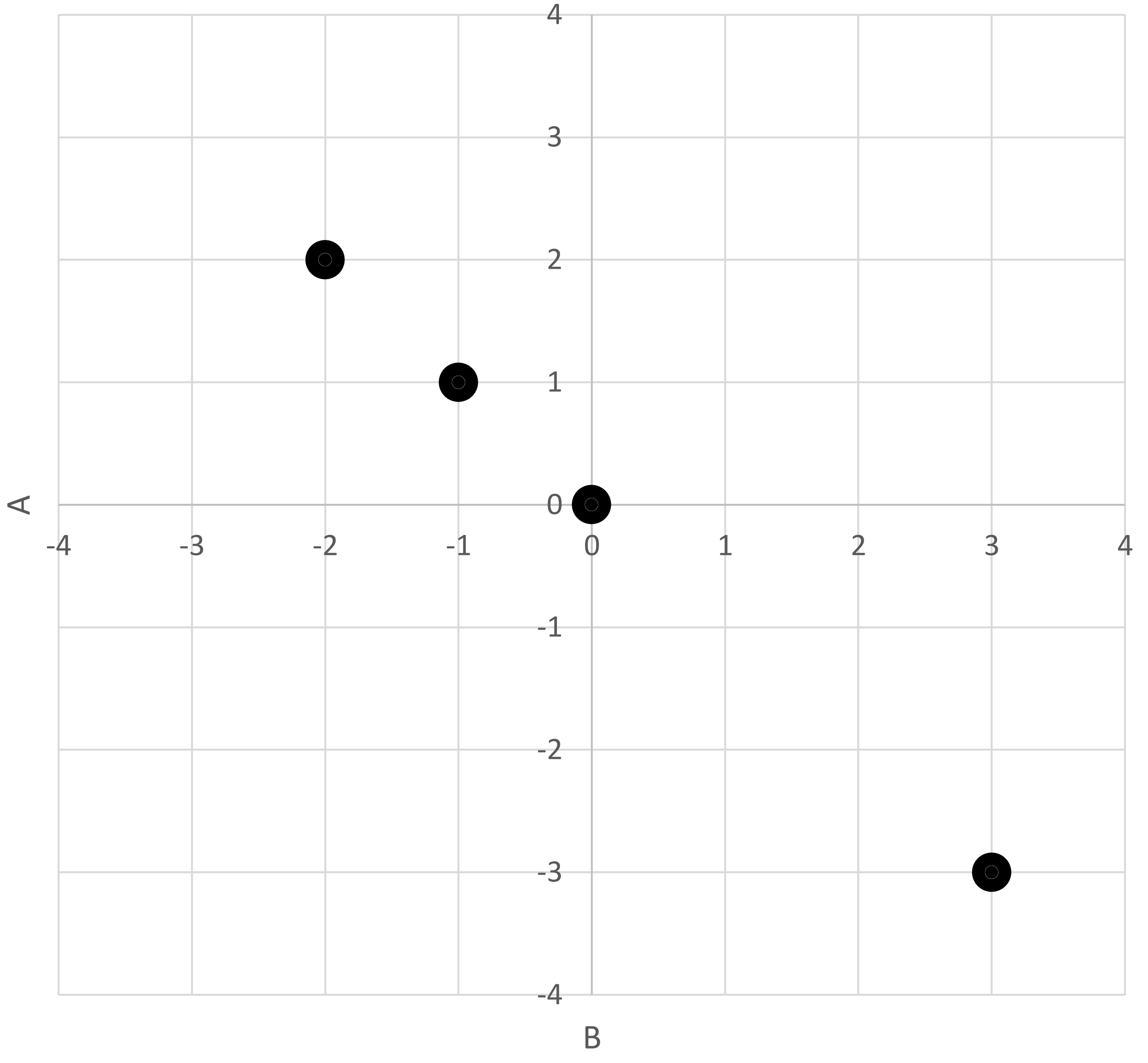
Game Y – Pareto efficiency

B

A

	L	R
U	3 , -3	-2 , 2
D	-1 , 1	0 , 0

Game Y – Pareto efficiency?



Game Y

		B: L (q)	B: R (1 - q)
A	U (p)	3 , -3	-2 , 2
A	D (1 - p)	-1 , 1	0 , 0

Game Y – Player A

- Player A plans to mix Up and Down strategy at a certain ratio p
- Player B might play Left or Right
- Player A must find such a **probability** of playing U and D that makes Player B **indifferent** to selecting L or R
- Player B **has to gain same utility** from B's choice Left and Right
  - $EU_L = EU_R$
- Expected utility of Player B chosing Left:
  - $EU_L = f(p)$
- Expected utility of Player B chosing Right:
  - $EU_R = f(p)$

Game Y

		B: L	B: R
A	U (p)	3 , -3	-2 , 2
A	D (1 - p)	-1 , 1	0 , 0

Game Y

		B	
		L	R
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

Game Y - Player A's strategy

- EU_L:
  - Some % of time (p) gets B utility -3
  - Rest of the time (1 - p) gets B utility 1

- $EU_L = (p)*(-3) + (1 - p)*(1)$
- $EU_L = -3p + 1 - p$
- $EU_L = 1 - 4p$

B

		L (q)	R (1 - q)
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

Game Y - Player A's strategy

- EU_R:
  - Some % of time (p) gets B utility 2
  - Rest of the time (1 - p) gets B utility 0

- $EU_R = (p)*(2) + (1 - p)*(0)$
- $EU_R = 2p + 0 - 0p$
- $EU_R = 2p$

B

A		L (q)	R (1 - q)
	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

Player A's strategy – making B indifferent Comparison of EU_L with EU_R

- $EU_L = 1 - 4p$
- $EU_R = 2p$

- $EU_L = EU_R$
- $1 - 4p = 2p$ +4p
- $1 = 6p$ /6
- **p = 1/6**

- **1 - p** = 1 - 1/6 = **5/6**

- We've found the ideal mixed strategy for Player A
- If Player A plays Up 1/6 of time and Down 5/6 of time, Player B is indifferent to choosing Left or Right
- We need to do the same for player B

Game Y

B

A		L (q)	R (1 - q)
	U	3 , -3	-2 , 2
	D	-1 , 1	0 , 0

Game Y - Player B’s strategy

- EU_U:
  - Some % of time (q) gets A utility 3
  - Rest of the time (1 - q) gets A utility -2

- $EU_U = (q)*(3) + (1 - q)*(-2)$
- $EU_U = 3q - 2 + 2q$
- $EU_U = 5q - 2$

B

		L (q)	R (1 - q)
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

Game Y - Player B's strategy

- EU_D:
  - Some % of time (q) gets A utility -1
  - Rest of the time (1 - q) gets A utility 0

- $EU_D = (q)*(-1) + (1 - q)*(0)$
- $EU_D = -1q + 0 - 0q$
- $EU_D = -q$

B

A	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

Player B's strategy – making A indifferent Comparison of EU_U with EU_D

- $EU_U = 5q - 2$
- $EU_D = -q$

- $EU_U = EU_D$
- $5q - 2 = -q$ $-5q$
- $-2 = -6q$ $/-6$
- **$q = 1/3$**

- **$1 - q$** $= 1 - 1/3 =$ **$2/3$**

- We've found the ideal mixed strategy for Player B
- If Player B plays Left 1/3 of time and Down 2/3 of time, Player A is indifferent to choosing Up or Down

Mixed strategy NE

(1/6 U , 1/3 L)

Game Y - MSNE

B

A		L (1/3)	R (2/3)
	U (1/6)	3 , -3	-2 , 2
	D (5/6)	-1 , 1	0 , 0

Battle of sexes

- Want to go out together but have no means of communication
- Have 2 choices – ballet or car show
- Player A prefers car show (C)
- Player B prefers ballet (B)
- Both prefer being together than being alone (A)

- Preferences for player A: C > B > A
- Preferences for player B: B > C > A

Battle of sexes

		B	
		b	c
A	B	1 , 2	0 , 0
	C	0 , 0	2 , 1

Battle of sexes – PS Nash equilibriums

		B	
		b	c
A	B	1 , 2	0 , 0
	C	0 , 0	2 , 1

Equilibriums

- 2 pure-strategies equilibriums
- How would they coordinate?
- Apart from pure strategies equilibriums there is one mixed strategy equilibrium for this game
- (1/3 B, 2/3 b)

Battle of sexes – mixed strategy equilibrium

B

A		b 2/3	c 1/3
	B 1/3	1 , 2	0 , 0
	C 2/3	0 , 0	2 , 1

Calculation of MS NE payoffs

Battle of sexes – mixed-strategy NE payoffs

B

		b 2/3	c 1/3
A	B 1/3	1 , 2 1/3 * 2/3	0 , 0 1/3 * 1/3
	C 2/3	0 , 0 2/3 * 2/3	2 , 1 2/3 * 1/3

Battle of sexes – mixed-strategy NE payoffs

		B	
		b 2/3	c 1/3
A	B 1/3	1 , 2 2/9	0 , 0 1/9
	C 2/3	0 , 0 4/9	2 , 1 2/9

BoS – Payoffs for player A

- We simply multiply payoffs for player A and probabilities for each outcome and then sum them together

- Player A's payoffs:
  - u(B, b) = 1 * 2/9 = 2/9
  - u(B, c) = 0 * 1/9 = 0
  - u(C, b) = 0 * 4/9 = 0
  - u(C, c) = 2 * 2/9 = 4/9

- EU(A) = 2/9 + 0 + 0 + 4/9
- EU(A) = 6/9
- EU(A) = 2/3

B

A

	b 2/3	c 1/3
B 1/3	1 , 2 2/9	0 , 0 1/9
C 2/3	0 , 0 4/9	2 , 1 2/9

BoS – Payoffs for player B

- We simply multiply payoffs of player B and probabilities for each outcome and then sum them together

- Player A's payoffs:
  - u(B, b) = 2 * 2/9 = 4/9
  - u(B, c) = 0 * 1/9 = 0
  - u(C, b) = 0 * 4/9 = 0
  - u(C, c) = 1 * 2/9 = 2/9

- EU(B) = 4/9 + 0 + 0 + 2/9
- EU(B) = 6/9
- EU(B) = 2/3

B

A

	b 2/3	c 1/3
B 1/3	1 , 2 2/9	0 , 0 1/9
C 2/3	0 , 0 4/9	2 , 1 2/9

Battle of sexes NE

- Pure strategies NE
  - (B , b)
    - EU(A) = 1
    - EU(B) = 2
  - (C , c)
    - EU(A) = 2
    - EU(B) = 1
- Mixed strategies NE
  - (1/3 B , 2/3 b)
    - EU(A) = 2/3
    - EU(B) = 2/3

B

A		b	c
	B	1 , 2	0 , 0
	C	0 , 0	2 , 1

FSS entrance game

- Two students meet at the main faculty entrance
- Both simultaneously decide whether to walk or stop
- If both walk, they **collide** and both **get a bruise** (payoff -5)
- If one stops and other walks
  - Student who stopped gets **good karma** for letting the other pass with **payoff 1**, but at the same time gets **delayed**, which is **completely offsetting** the value of the good karma
  - Student who walked gets to **pass quickly** and thus gets **payoff 1**
- If **both stop**, each would get **good karma** for letting the other pass, but botch will get **delayed**

FSS entrance game

A \ B	W	s
W	-5 , -5	1 , 0
S	0 , 1	0 , 0

FSS entrance game NE

		B: w 1/6	B: s 5/6
A	W 1/6	-5 , -5	1 , 0
A	S 5/6	0 , 1	0 , 0

Stag hunt

B

A		s	r
	S	5 , 5	0 , 3
	R	3 , 0	3 , 3

Stag hunt NE

- Pure strategies NE
  - (S , s)
    - EU(A) = 5
    - EU(B) = 5
  - (R , r)
    - EU(A) = 3
    - EU(B) = 3
- Mixed strategies NE
  - (3/5 S , 3/5 s)
    - EU(A) = 3
    - EU(B) = 3

B

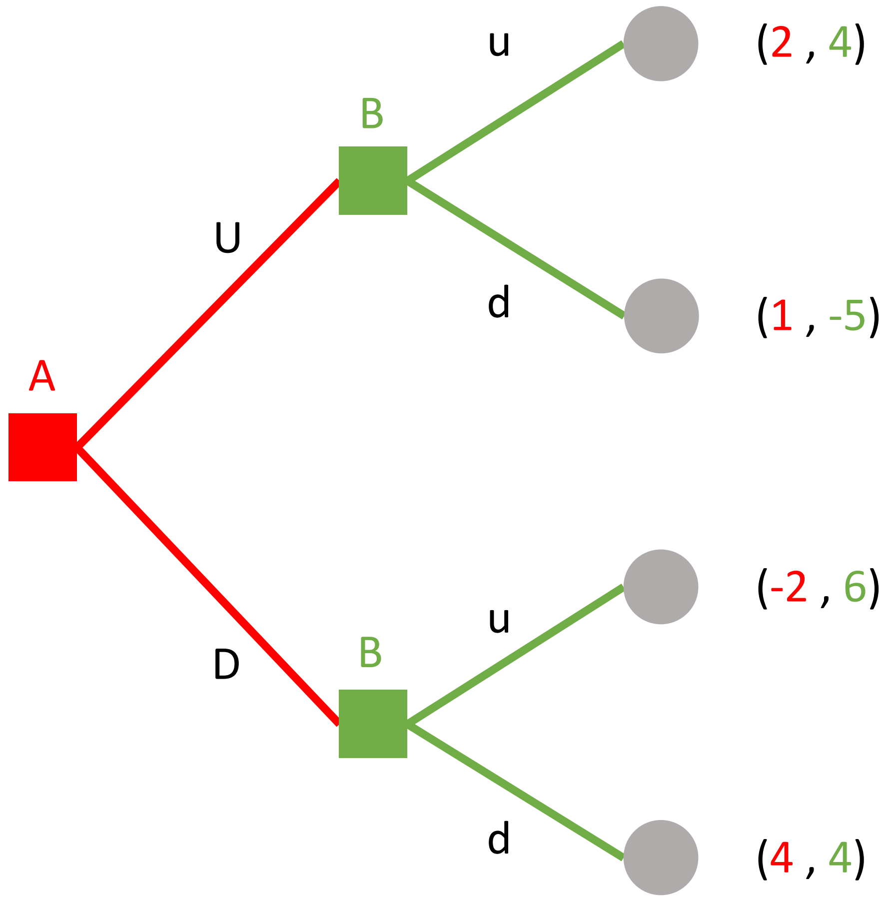
A

	s	R
S	5 , 5	0 , 3
R	3 , 0	3 , 3

Extensive form games

Extensive form games

- Visualized as a **game (decision) tree**
- Players move **sequentially**
- Captures **time** in game
- Captures knowledge of agents – sometimes agents do not have information where they are located in game

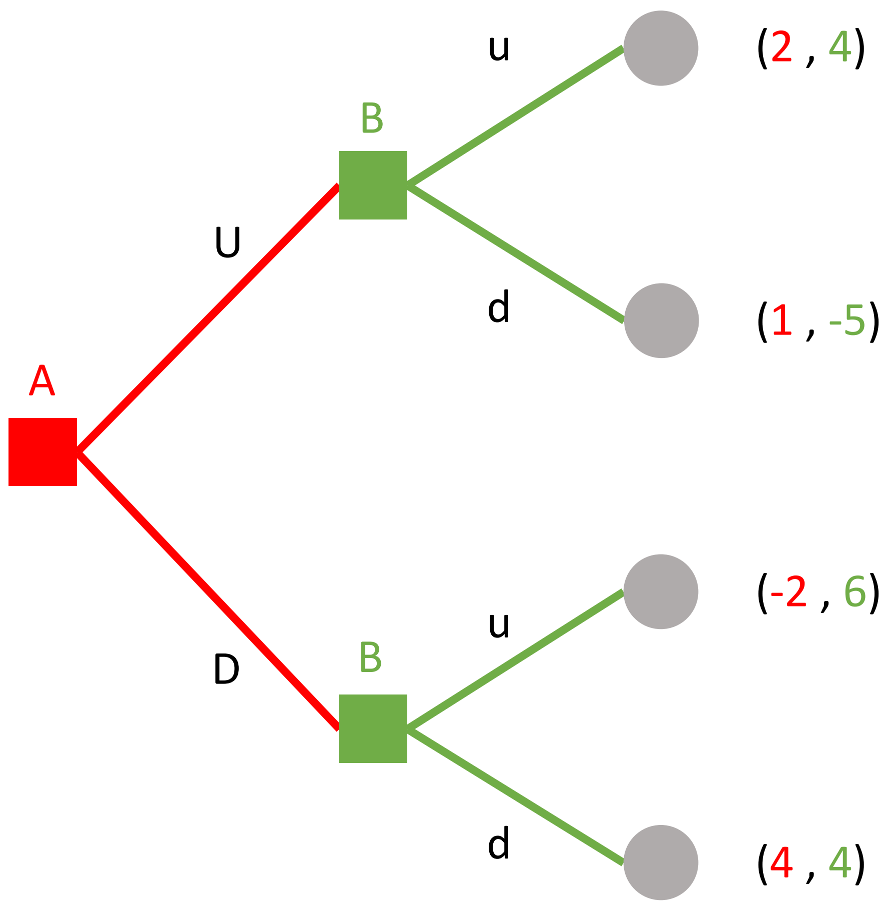


A
U
D
B
B
u
d
u
d
(2 , 4)
(1 , -5)
(-2 , 6)
(4 , 4)

Basic terminology

- Each square is called **node**
- Each line represents an **action** an **owner of the node** has at his disposal
- Nodes might either **trigger other action** or **end**
- Every circle is an end of the tree – it’s called **terminal node**
- Every circle must yield **payoffs** for all the actors
- Each moment actor has **information about all previous moves** called **information set**

Backwards induction

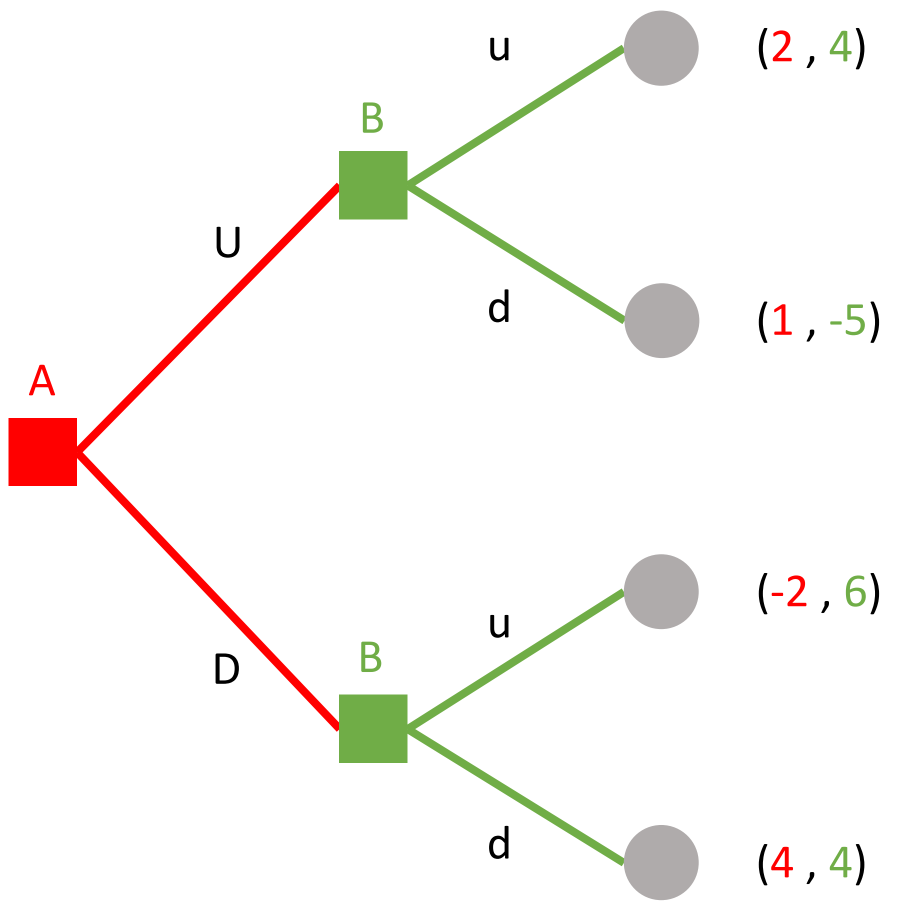

A
U
D
B
B
u
d
u
d
(2 , 4)
(1 , -5)
(-2 , 6)
(4 , 4)

Backwards induction

- Moves player make at nodes reached in an equilibrium are called **behavior on the equilibrium path**
- Moves player make at nodes that are not reached in an equilibrium are behavior **off the equilibrium path**

Backwards induction

- Begin with decisions that lead only to terminal nodes
- Compare payoffs for decisions in each node leading to terminal node
- Find **best reply to alternatives of player** playing at the **current node**
- Work through nodes **backwards** and solve the outcomes of all nodes comparing payoffs for respective players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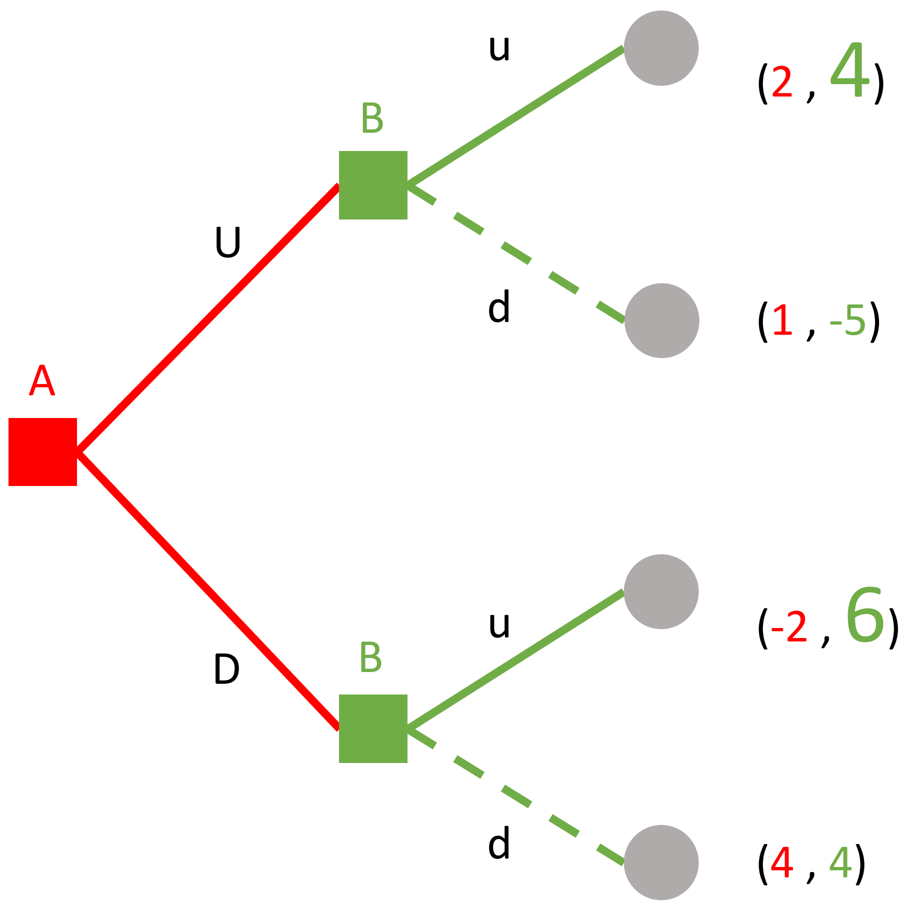
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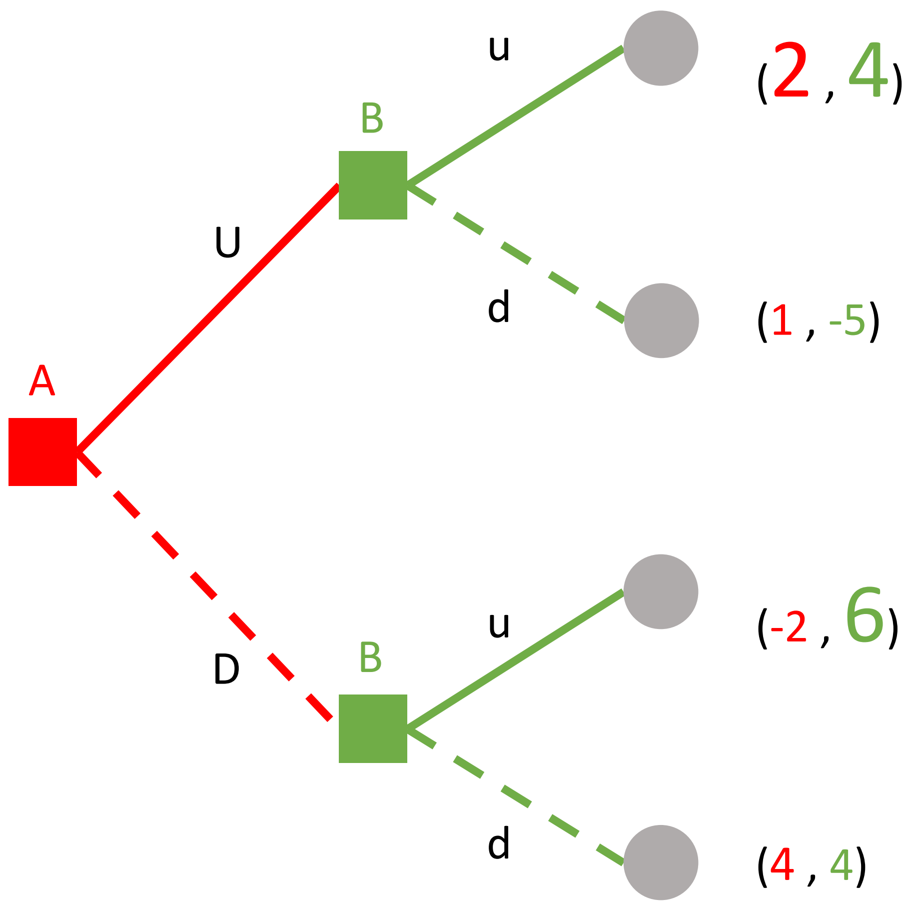

A
U
D
B
B
u
d
u
d
(2 , 4)
(1 , -5)
(-2 , 6)
(4 , 4)

Equilibrium of sequential game

- One Nash equilibrium in pure strategies on the equilibrium path
- (U ; u , u)

- There may be **more NE in pure strategies**

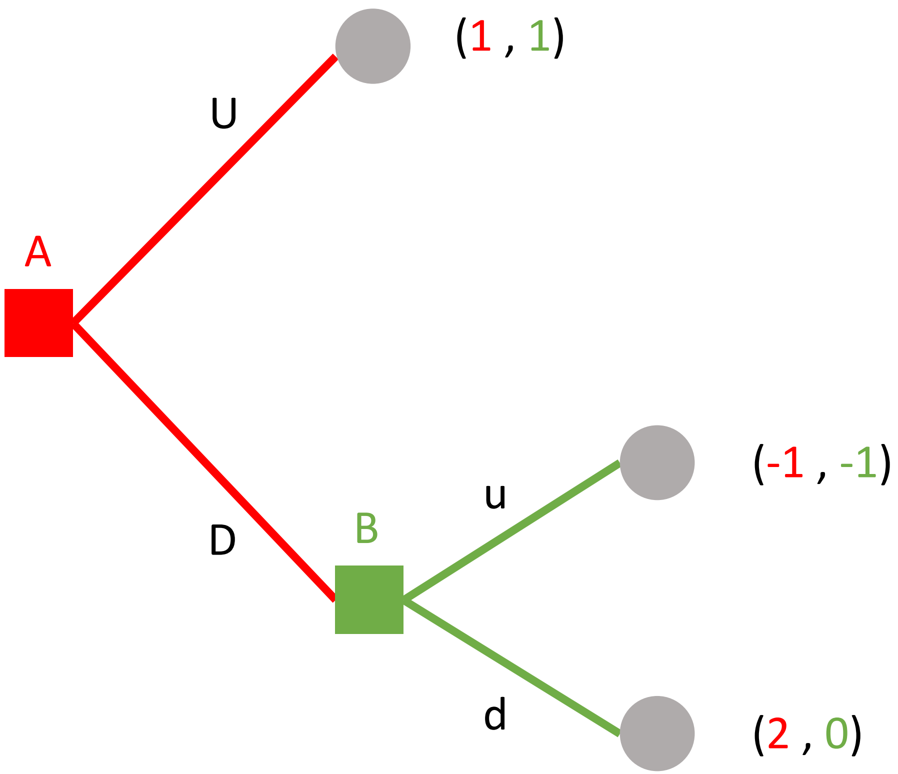
- B decides – if A goes U than u yields better payoff in the upper node, if A goes D than u yields better payoff in the lower node

- A knows that B will choose u in both nodes, therefore compares payoffs in u for going U or D – U yields better payoff

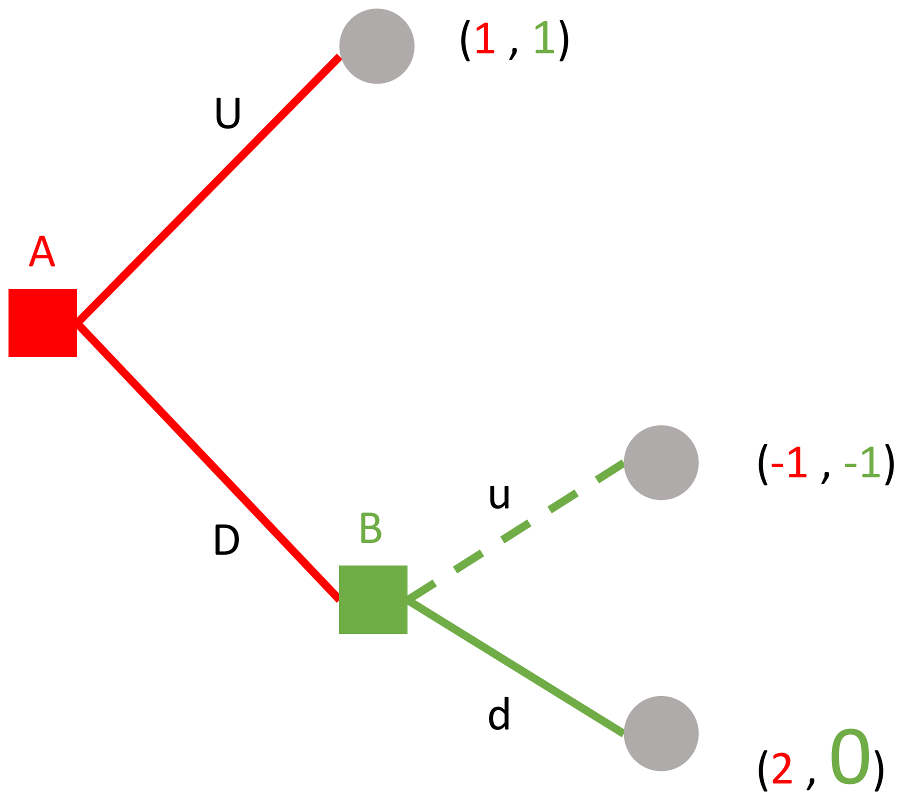
Backwards induction

- Works in games of **perfect information**
- All actors are aware of all previous actions and can also **anticipate**, what actors will do based on their expected utilities over outcomes at subsequent nodes – actors have a **perfect recall**
- However, backwards induction assesses only rationality on the equilibrium path. **NE found off the equilibrium path** will **not be found** through backwards induction

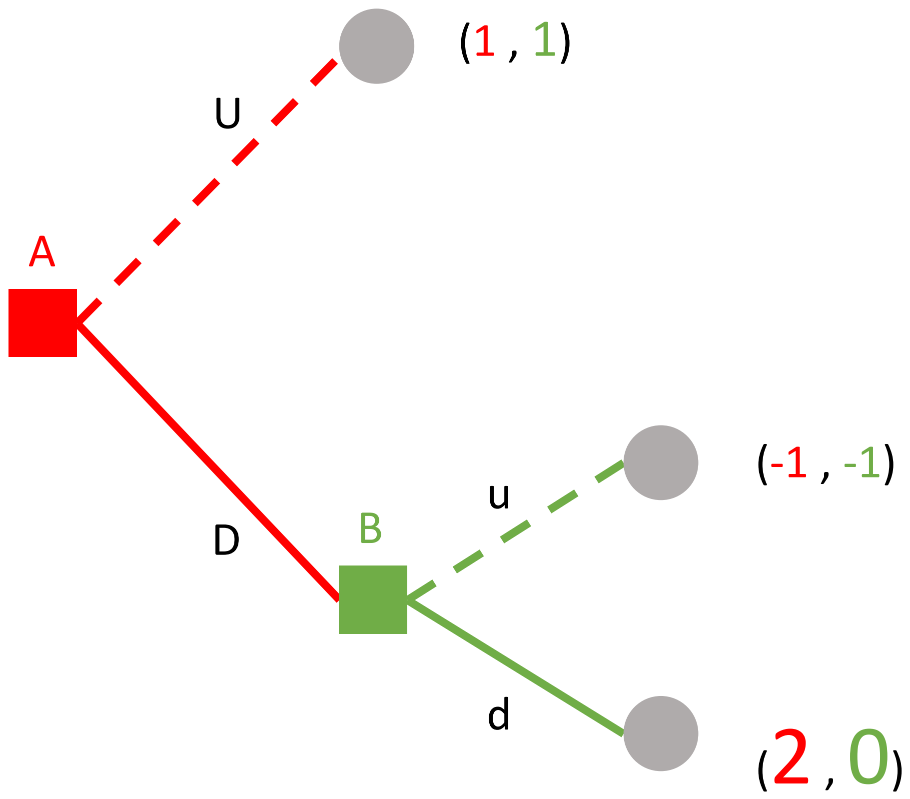
Selten's game



Selten's game



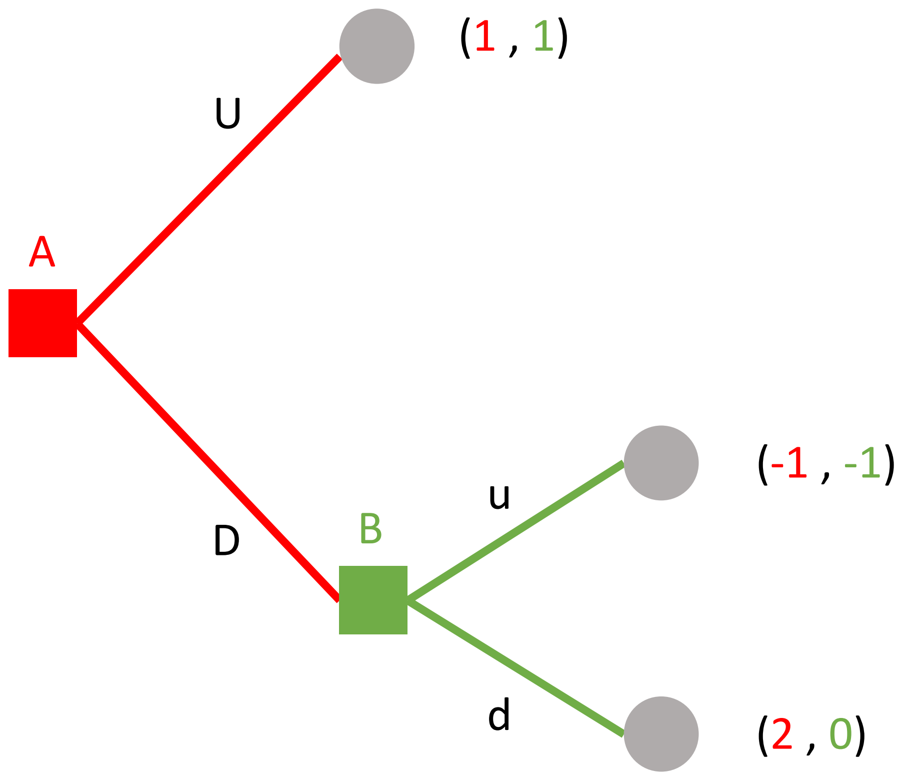
Selten’s game



Rewrite Selten's game into matrix

- Player A has 2 moves
  - U, D
  - Represented as 2 rows in a matrix
- Player B has also 2 moves
  - u, d
  - Represented as 2 columns in a matrix
- We have 2x2 game in normal form

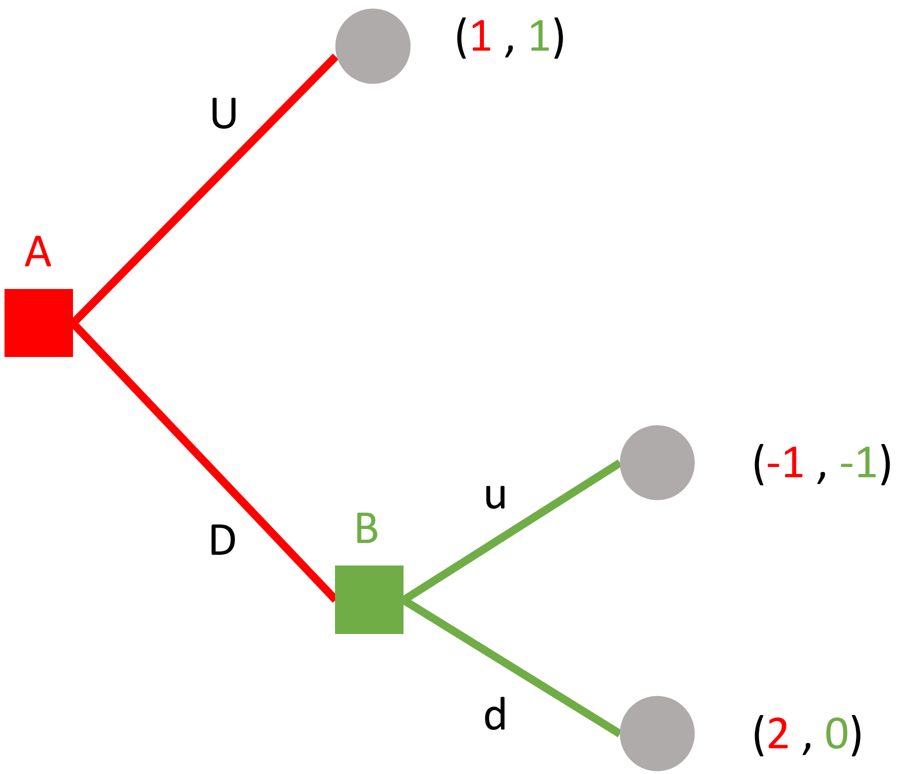
Selten's game



Strategic form of Selten’s game

		B	
		u	d
A	U		
	D		

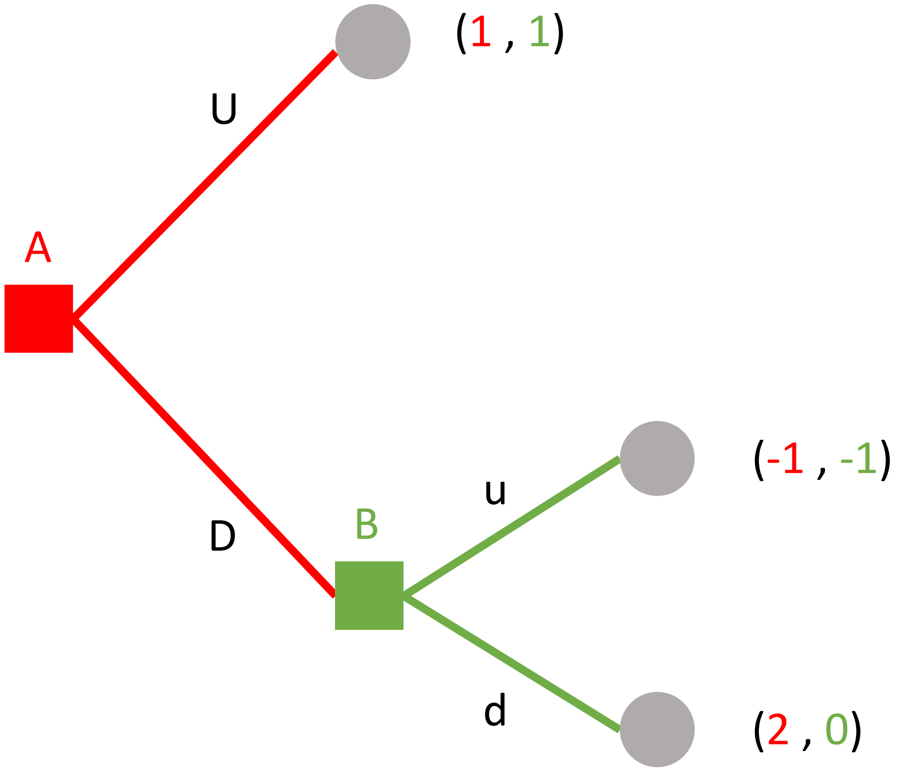
Selten’s game



Strategic form of Selten's game

		B	
		u	d
A	U		
	D	-1 , -1	2 , 0

Selten's game



Strategic form of Selten's game

B

A		u	d
	U	1 , 1	
	D	-1 , -1	2 , 0

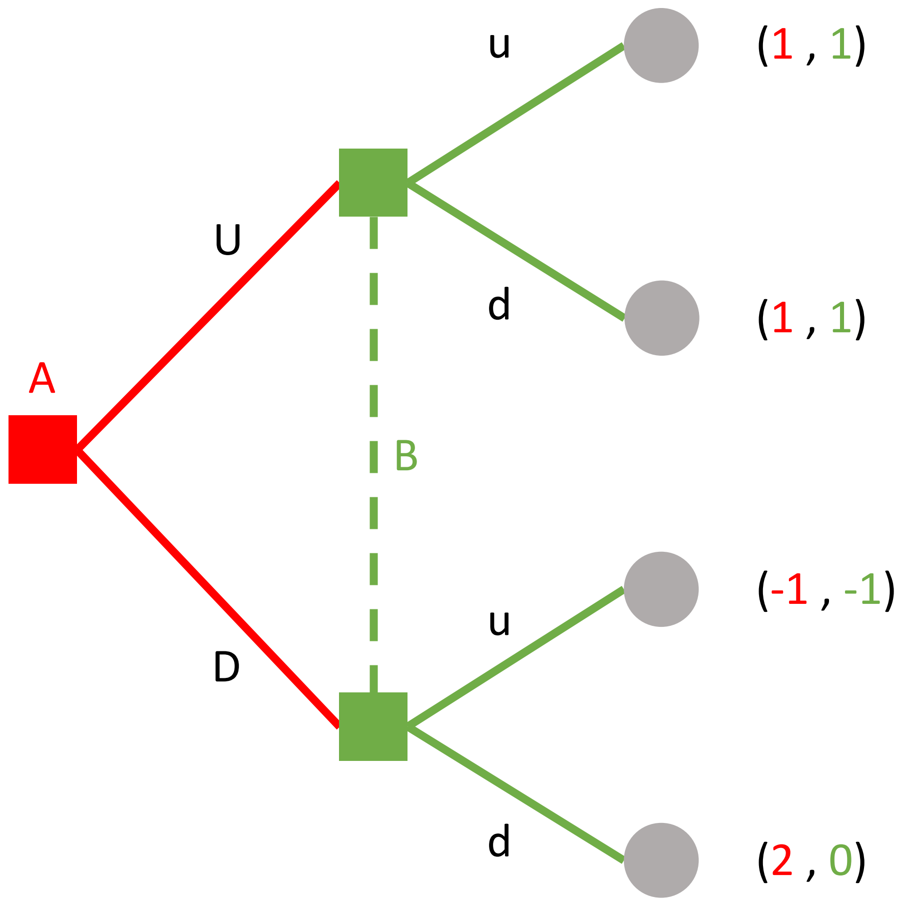
Strategic form of Selten's game

		B	
		u	d
A	U	1 , 1	1 , 1
	D	-1 , -1	2 , 0

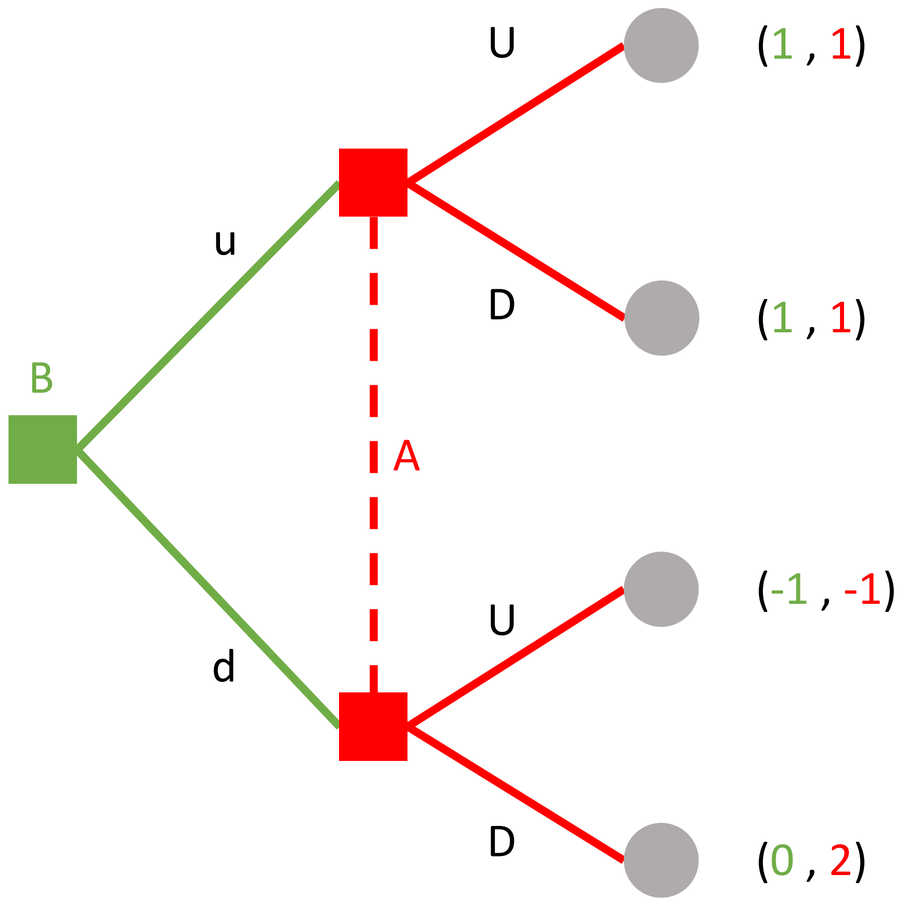
Rewriting extensive form into normal form

- If we rewrite extensive form game into normal form, only **one matrix will emerge** as a representation
- This **does not work the other way around**
- Since matrixes do not hold information about sequence of actions, one normal-form game might have multiple extensive-form representations that would completely alter outcomes of the game

Selten’s game from matrix



Selten's game from matrix



Selten's game – Nash equilibrium

		B: u	B: d
A	U	1 , 1	1 , 1
A	D	-1 , -1	2 , 0

Selten's game – Nash equilibrium

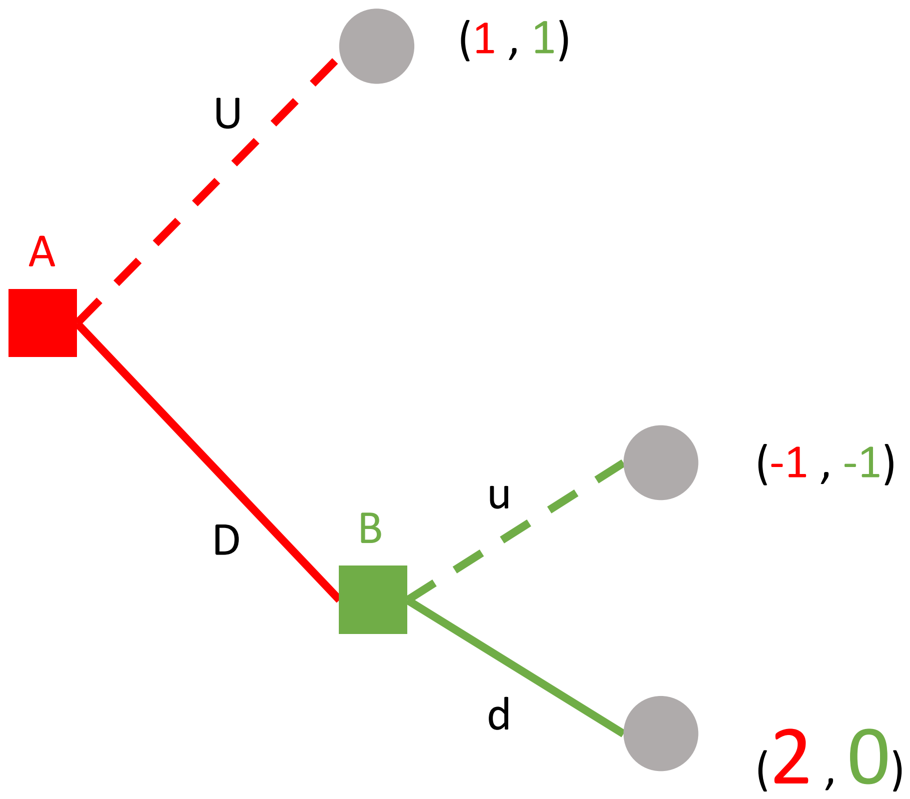
		B	
		u	d
A	U	**1** , 1	1 , 1
	D	-1 , -1	**2** , 0

Selten’s game – Nash equilibrium

B

	u	d
A: U	1 , 1	1 , 1
A: D	-1 , -1	2 , 0

Selten's game



NE off the equilibrium path

- This game has another NE which is represented by action U which is not revealed in extensive form
- This equilibrium (U, u) is called a **non-credible threat**
  - B is willing to get its largest payoff, which will result from A playing U (deciding not to play the game)
  - B can change its payoff from A’s decision about U and D only by threatening that it will play u if A plays D
  - But B will never play u, since it brings lower payoff than playing d

Selten's game and non-credible threat