Game theory 1

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MEB421

Agent vs. structure

- On the **individualist extreme** of agent – structure spectrum
- Regards the social phenomena as interactions of individuals – **structure is a product** of individual behavior
- Individual behavior is **unconstrained**

Criticism against rational choice theory

- Common criticism of rational choice – people behave irrationally
- Rationality ≠ Sensibility
- Ordering preferences
- I can prefer taking over the world over painful death, but equally prefer painful death over taking over the world

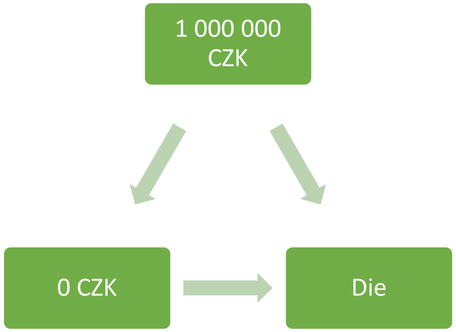
Rationality

- Defined by two key premises
  - Completeness
  - Transitivity

- **Indifferent to normative assessment** of preferences and choices

Completeness

- Preference ordering complete if and only if for any two outcomes X and Y individual:
    - A) Prefers X to Y – strong preference relation
    - B) Prefers Y to X – strong preference relation
    - C) Is indifferent – weak preference relation



1 000 000 CZK

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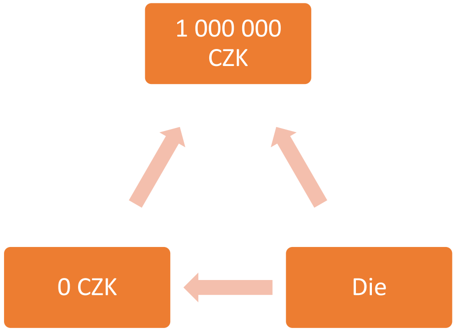
Die



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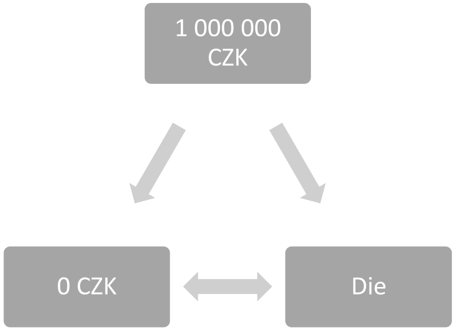

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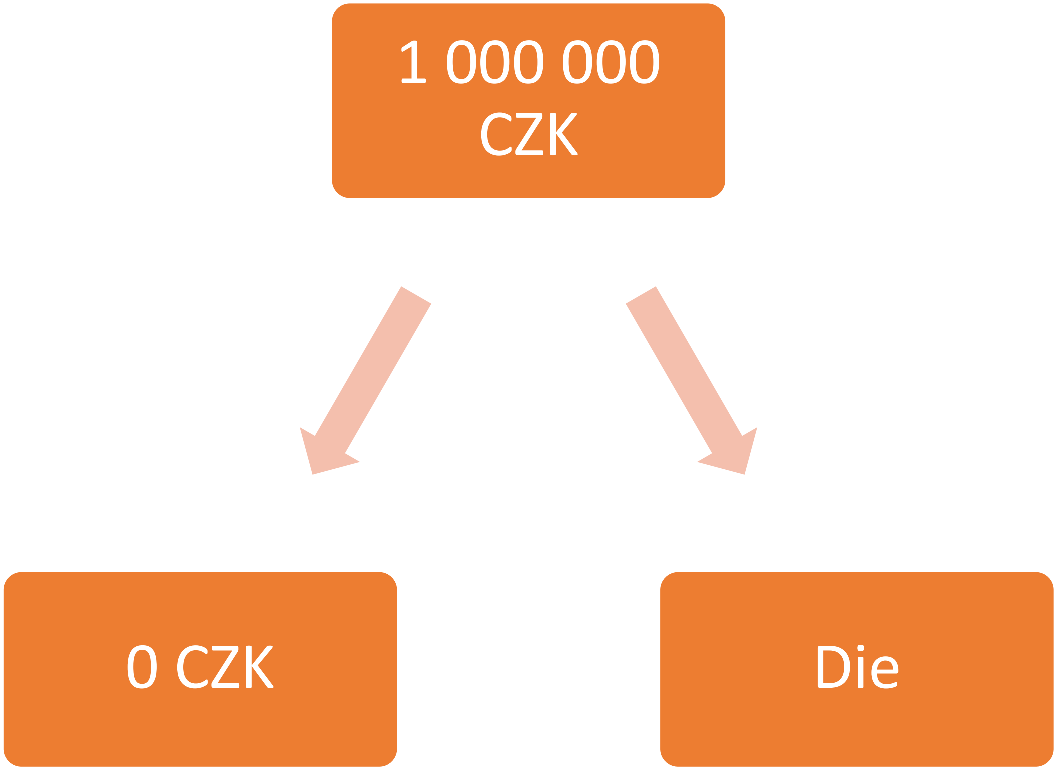
Die

Incomplete preferences



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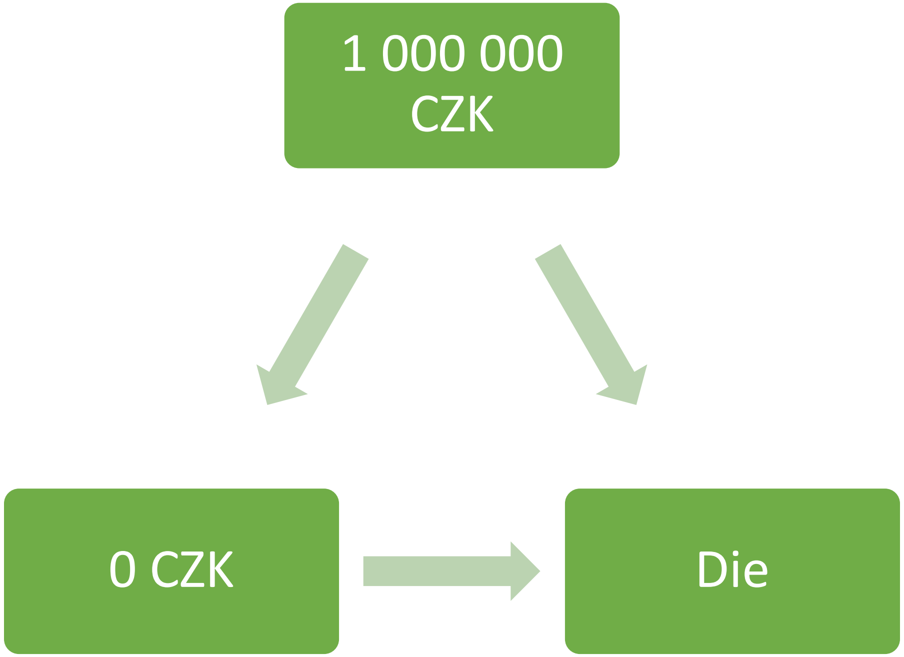
Die

Transitivity

- For any three outcomes X, Y and Z, if X is preferred to Y and Y is preferred to Z, X must be preferred to Z



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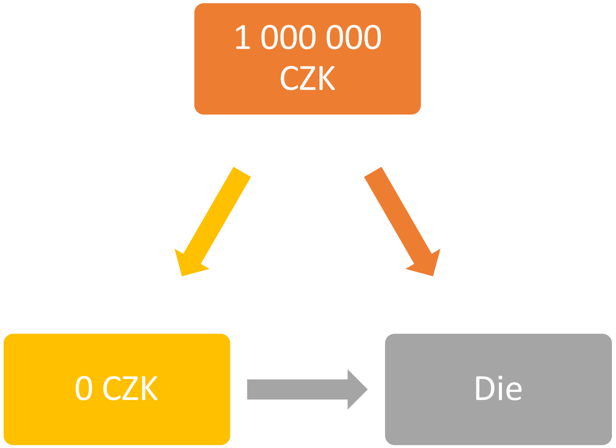

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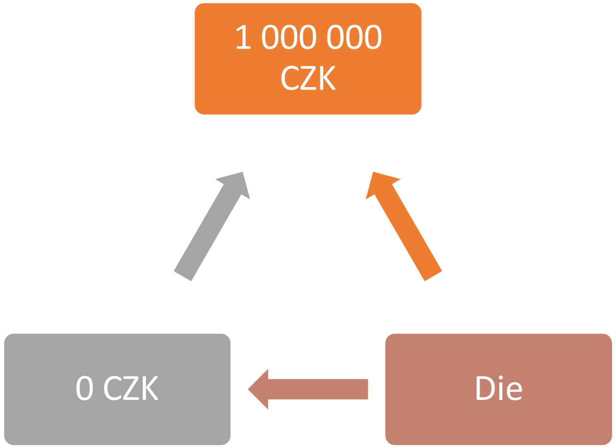

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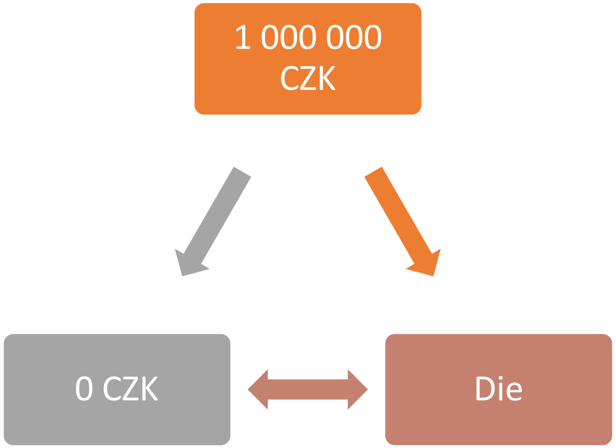

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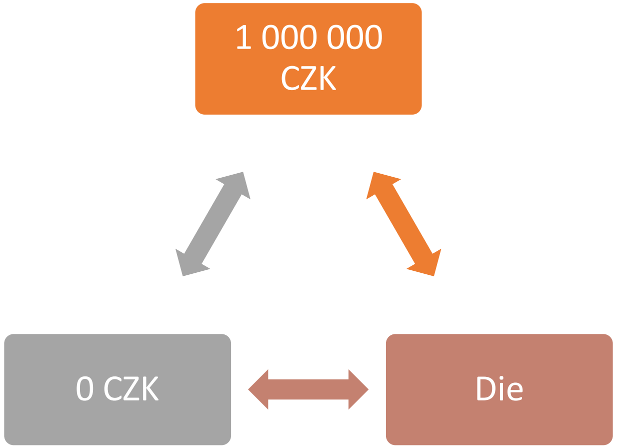
Die



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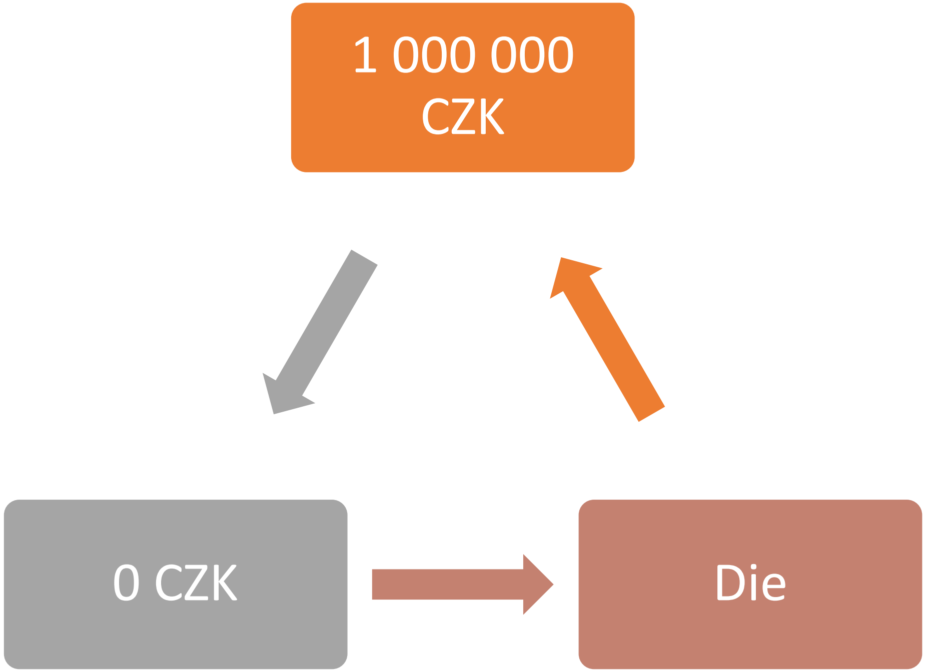
Die

Intransitive preferences

- Prefer X to Y, Y to Z and Z to X
- Doesn’t make sense



1 000 000 CZK

0 CZK

Die

Other notions about preferences

- Preferences over outcomes are **stable** and do **not change in the time of making decision** – are fixed

- Preferences are **ordinal** – they order actions but the difference between the two values has **no meaning unless they state utility**

- Compare two situations
  - $u(C_1) = 1$, $u(C_2) = 2$, $u(C_3) = 0$
  - $u(C_1) = 1$, $u(C_2) = 200$, $u(C_3) = -50$

- Both situations have same preference ordering
  - C_2 p C_1 p C_3

Other notions about rationality

- Rational choice theory is not attempting to explain cognitive processes happening in individuals

- Rationality tells **nothing about preferences over outcomes**

- Rational actors may **differ in choices in same situation**

- Rational actors can **err**

Types of games

Types of games

- Games of perfect information
- Games of imperfect information

- Cooperative games
- Non-cooperative games

- Constant-sum game
- Positive-sum game

Games of perfect/imperfect information

Perfect information games

- All players know other players' strategies available to them
- All players know payoffs over actions
- All players know what other players know

Imperfect information games

- Some information about other players' actions is not known to a player

Cooperative/non-cooperative games

Cooperative games

- Actors are allowed to make enforceable contracts
- Players do not need to cooperate, but cooperation is enforceable by an outside party

Non-cooperative games

- Actors unable to make enforceable contracts outside of those specifically modelled in the game
- Players might cooperate, but any cooperation must be self-enforcing

Constant-sum/Positive-sum games

Constant sum games

- Sum of all players' payoffs is the same for any outcome
- Gain for one participant is always at the expense of another
- Special case of zero-sum game where all outcomes involve a sum of all player's payoffs of 0

Positive-sum games

- Combined payoffs of all players are not the same in every outcome of the game
- Positive-sum game implies that players may have interests in common, to achieve an outcome that maximizes total payoffs.

Introducing a game

What makes a game the game

- Players
- Actions
- Strategies
- Outcomes
- Payoffs of player

Game of grades

- Each pair can choose 2 actions: α or β
- If both choose α, both will receive **C**
- If both choose β, both will receive **B**
- If one chooses α and other β, one will receive **A** and other **D**

Game of grades – my grades

My opponent

Me

	α	β
α	C	**A**
β	**D**	B

Game of grades – my opponent's grades

My opponent

		α	β
Me	α	C	**D**
	β	**A**	B

Game of grades – normal form

		My opponent	
		α	β
Me	α	C , C	A , D
	β	D , A	B , B

Games in normal form

Normal form representation of a game

- Called also “strategic form” or “matrix form”
- Visualized as a **matrix**
- Represents a game as if agents were acting **simultaneously**

Utilities (Payoffs)

- Grades are not **utilites**

- Utilities for game:
  - EU(A) = 3
  - EU(B) = 2
  - EU(C) = 1
  - EU(D) = 0

- Preference over outcomes: A > B > C > D -> APBPCPD

Game of grades with payoffs

My opponent

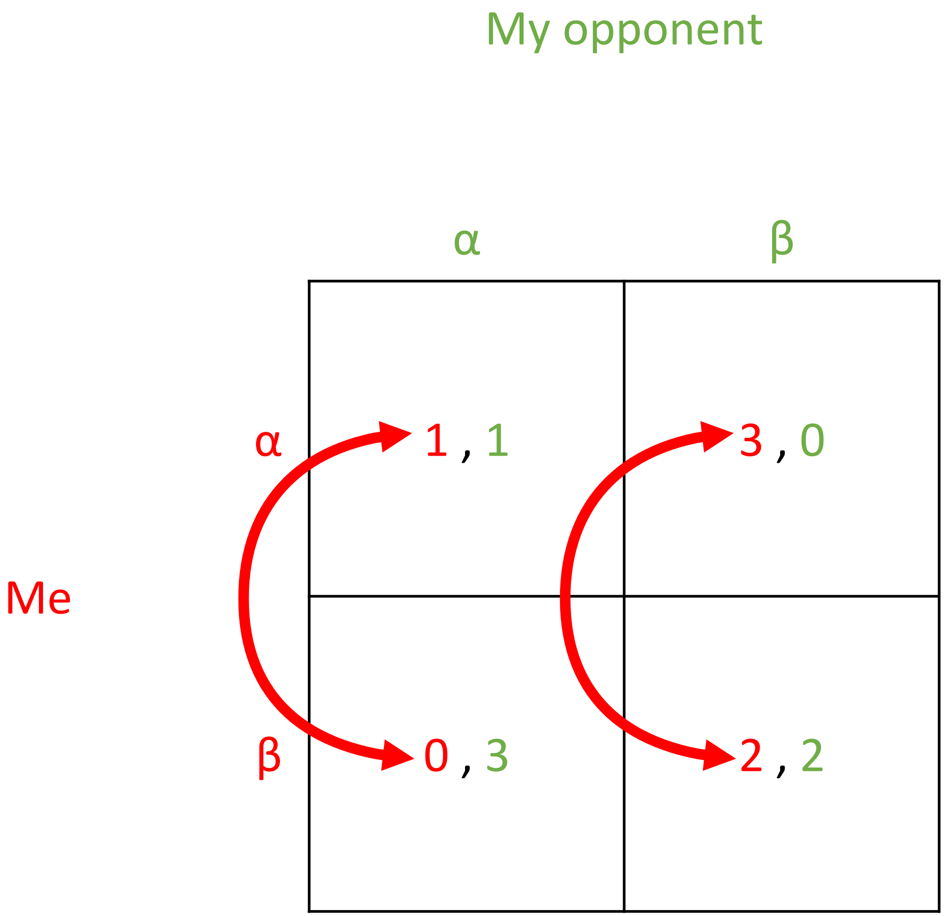
		α	β
Me	α	1 , 1	3 , 0
	β	0 , 3	2 , 2

Solution concepts

- Nash Equilibrium
  - Dominant Strategy Equilibrium
  - Pure Strategy Equilibrium
  - Mixed Strategy Equilibrium
- Subgame Perfect Equilibrium
- Bayesian Equilibrium
- Weak Perfect Bayesian Equilibrium

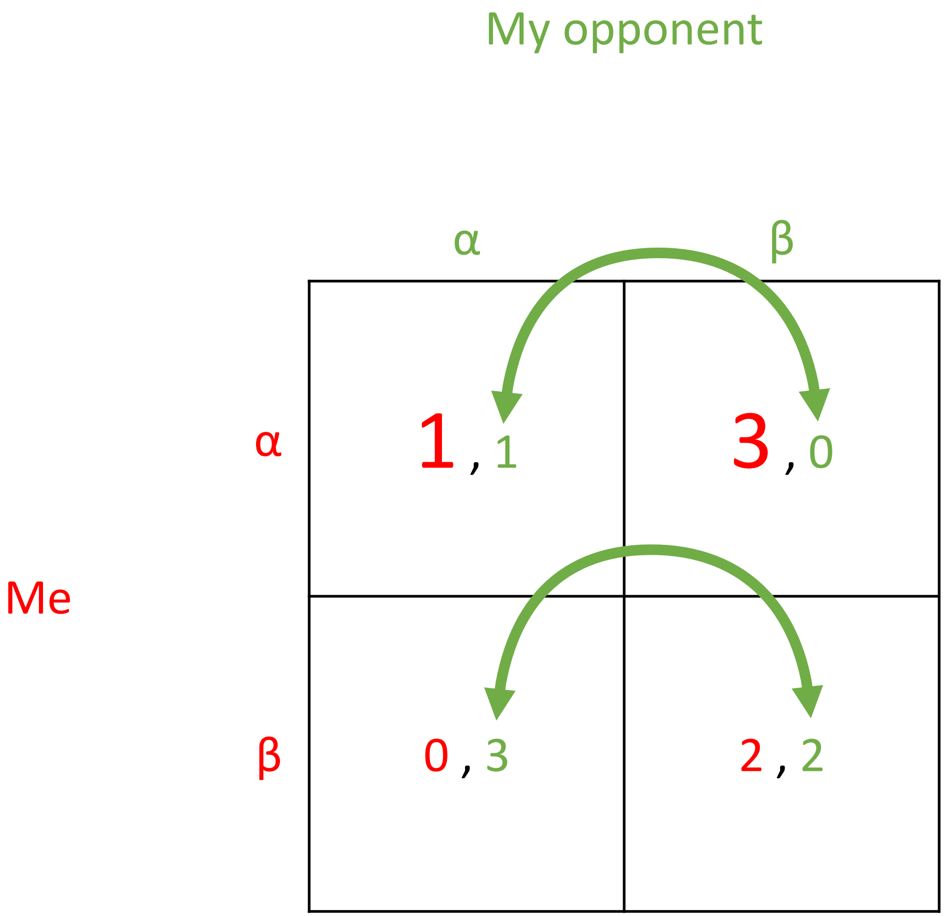
My opponent

		α	β
Me	α	1 , 1	3 , 0
	β	0 , 3	2 , 2



My opponent

		α	β
Me	α	1 , 1	3 , 0
	β	0 , 3	2 , 2



My opponent

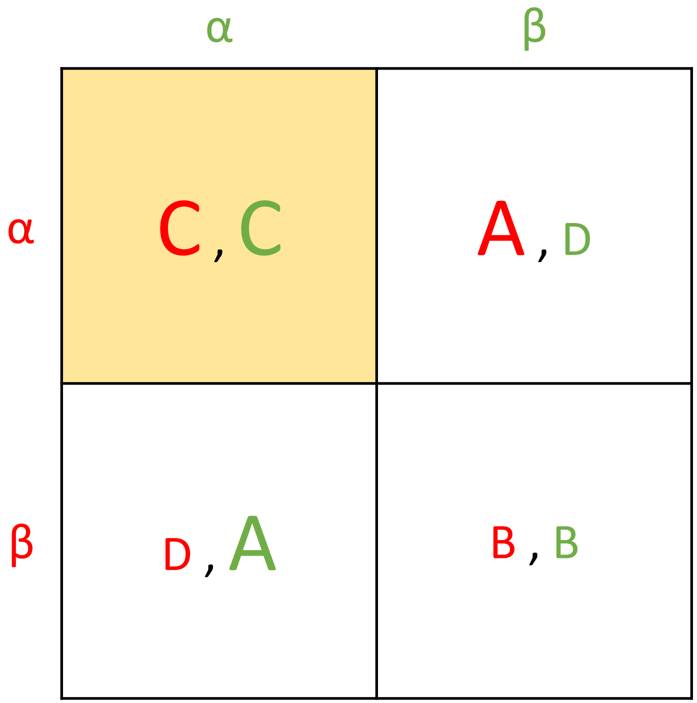
Me

	α	β
α	1 , 1	3 , 0
β	0 , 3	2 , 2

My opponent

		α	β
Me	α	1 , 1	3 , 0
	β	0 , 3	2 , 2

My opponent



		α	β
Me	α	C , C	A , D
	β	D , A	B , B

Prisoner’s dilemma

- Both players are tempted to **defect**, since cooperate is **strictly dominated** by defect
- The outcome of the game is that both players **betray** the other one and end up choosing α
- Both will end up with **outcome that is less preferred** than the “optimal” outcome β, β by seeking **maximal gain from own action**

Dominance

Dominant Strategy Equilibrium

- Strategy might be dominant

Two types of dominance

- Strict (strong) dominance
- Weak dominance

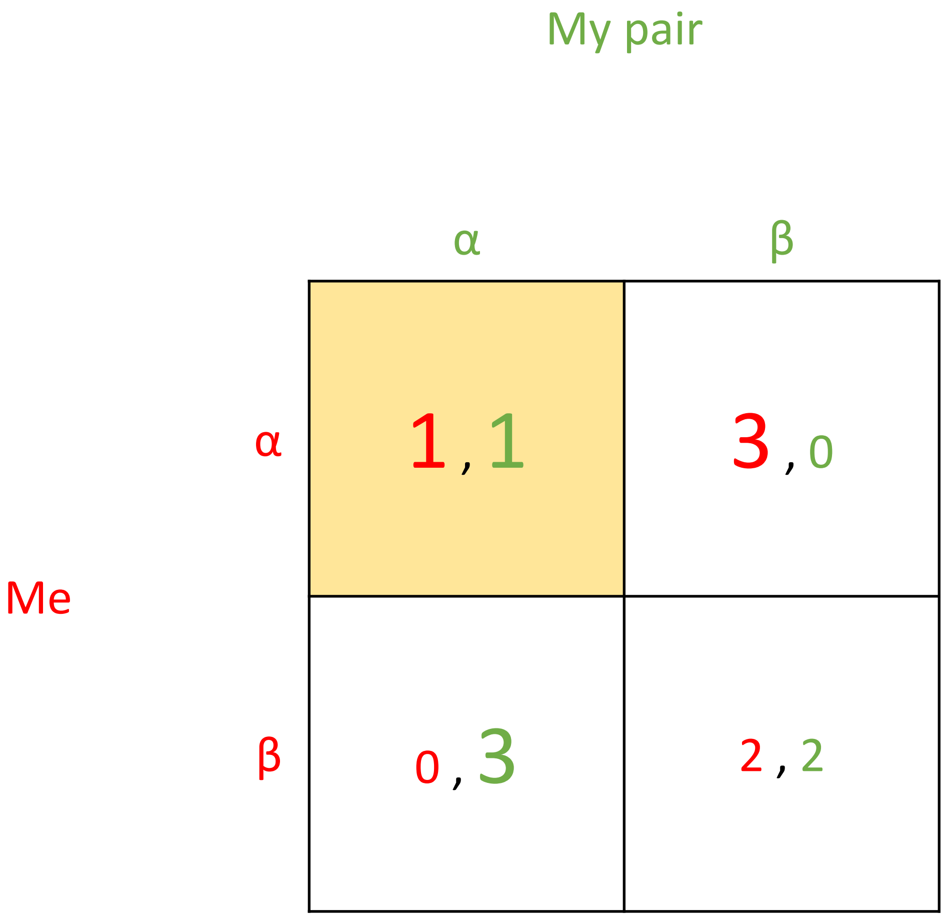
Strict dominance

- Player i
- Payoff u_i
- Dominant strategy s_i
- Dominated strategy s_i'
- Strategy of all other players s_{-i}

- Player i‘s strategy s_i' is strictly dominated by player i‘s strategy s_i if and only if
- $u_i(s_i , s_{-i})$ **>** $u_i(s_i' , s_{-i})$ for **all** s_{-i}

- utility of playing s_i against others’ strategies s_{-i} is **greater** than utility of playing s_i’ against others’s strategies s_{-i} for all others’ strategies s_{-i}

Game of grades – strict dominance



		My pair	
		α	β
Me	α	1 , 1	3 , 0
	β	0 , 3	2 , 2

Weak dominance

- Player i
- Payoff u_i
- Dominant strategy s_i
- Dominated strategy s_i'
- Strategy of all other players s_{-i}

- Player i‘s strategy s_i' is weakly dominated by player i‘s strategy s_i if
- $u_i(s_i , s_{-i}) \geq u_i(s_i' , s_{-i})$ for **all** s_{-i} and
- $u_i(s_i , s_{-i}) > u_i(s_i' , s_{-i})$ for **some** s_{-i}

- utility of playing s_i against others’ strategies s_{-i} is **greater or equal to** utility of playing s_i' against others’s strategies s_{-i} for all others’ strategies s_{-i} and **greater for some** others’ strategies s_{-i}

Game of grades – weak dominance

My pair

Me

	α	β
α	1 , 1	3 , 0
β	0 , 3	3 , 2

Never play dominated strategies

- Dominated strategy **brings lesser payoffs** than dominant strategy
- Dominated strategy brings lesser payoffs **no matter what strategy is selected by other player**
- Can’t control minds of others to force them not to play dominant strategy
- Event if **could** control minds of others and be sure they’ll play dominated strategy, than **rational to play dominant strategy anyway**

http://bit.ly/game-id

Choosing numbers

- Choose integer between 1 – 100 incl.
- All numbers will be averaged
- Winner is the one who will be closest to the 2/3 of the group's average

Choosing numbers

- Average = 100
- 2/3 of average = ~ 66.66
- X > 67 is strictly dominated strategy
  - Even if everyone else selected 100
  - One selected 67
  - I selected 68
  - Outcome – 68 is dominated by 67

- What is the rational choice for this game?

If all players were strictly rational, result is 1

I know you know

- I know
  - Numbers above 67 are never rational
- You know that I know
  - You’ll never select number above 67, therefore numbers above 46 are never rational either
- I know You know that I know
  - I know that You’ll never select above 46, hence I should never select number higher than 30
- You know that I know that You know that I know
  - You know that I won’t select above 30, therefore I should never select number above 20

Get into opponent’s shoes

Real life results

- 2012 Game theory online course

- 10 000 + players

- Mean 34
- Mode 50
- Median 33
- Winner 23

- Spikes: 50, 33, 20, 1

Iterated deletion of dominated strategies

Iterated deletion of dominated strategies

- Can delete dominated strategies as if they were not present in the game

- Game becomes simpler than the original one

- Can find equilibriums quickly – games are dominance-solvable

Game of grades

My pair

Me

	α	β
α	1 , 1	~~3 , 0~~
β	0 , 3	~~2 , 2~~

My pair

		α
Me	α	1 , 1
	β	~~0 , 3~~

My pair

		α
Me	α	1 , 1

This game is dominance-solvable

Opponent

		s_1	s_2	s_3
	S_1	0 , 1	-2 , 3	4 , -1
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

S_1 vs S_2

Opponent

		s_1	s_2	s_3
	S_1	0 , 1	-2 , 3	4 , -1
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

S_1 vs S_3

Opponent

		s_1	s_2	s_3
	S_1	0 , 1	-2 , 3	4 , -1
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

S_2 vs S_3

Opponent

		s_1	s_2	s_3
	S_1	0 , 1	-2 , 3	4 , -1
Me	S_2	0 , 3	3 , 1	**6** , 4
	S_3	**1** , 5	**4** , 2	5 , 2

S_1 VS S_3

Opponent

		s_1	s_2	s_3
Me	S_1	0 , 1	-2 , 3	4 , -1
	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

S_1 VS S_2

Opponent

		s_1	s_2	s_3
	S_1	0 , 1	-2 , 3	4 , -1
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

S_2 VS S_3

Opponent

		s_1	s_2	s_3
	S_1	0 , 1	-2 , 3	4 , -1
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

Opponent

		s_1	s_2	s_3
	S_1	~~0 , 1~~	~~-2 , 3~~	~~4 , -1~~
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

Opponent

		s_1	s_2	s_3
Me	S_2	0 , 3	3 , 1	6, 4
	S_3	1, 5	4 , 2	5 , 2

s_1 vs s_3 after deletion

Opponent

Me

	s_1	s_2	s_3
S_2	0 , 3	3 , 1	6 , 4
S_3	1 , 5	4 , 2	5 , 2

s_1 vs s_2 after deletion

Opponent

Me

	s_1	s_2	s_3
S_2	0 , 3	3 , 1	6 , 4
S_3	1 , 5	4 , 2	5 , 2

s_2 vs s_3 after deletion

		Opponent		
		s_1	s_2	s_3
Me	S_2	0 , 3	3 , 1	6 , 4
	S_3	1 , 5	4 , 2	5 , 2

Opponent

		s_1	s_2	s_3
Me	S_2	0 , 3	~~3 , 1~~	6 , 4
	S_3	1 , 5	~~4 , 2~~	5 , 2

Opponent

Me

	s_1	s_3
S_2	0 , 3	6 , 4
S_3	1 , 5	5 , 2

Opponent

Me

	s_1	s_3
S_2	0 , 3	6 , 4
S_3	1 , 5	5 , 2

Opponent

		s_1	s_3
Me	S_2	0 , 3	6 , **4**
	S_3	1 , **5**	5 , 2

Games sometimes not dominance solvable, but simplified

Limits of iterated deletion of dominated strategies

- **Strictly** dominated strategies may be deleted in a **random** order

- Deleting **weakly** dominated strategies in some order **might delete** equilibriums

- This solution concept is not always applicable – **sometimes game simply don’t have dominance**

How to solve the game without dominance?

		Opponent	
		s_1	s_3
Me	S_2	0 , 3	6 , 4
	S_3	1 , 5	5 , 2

Nash Equilibrium

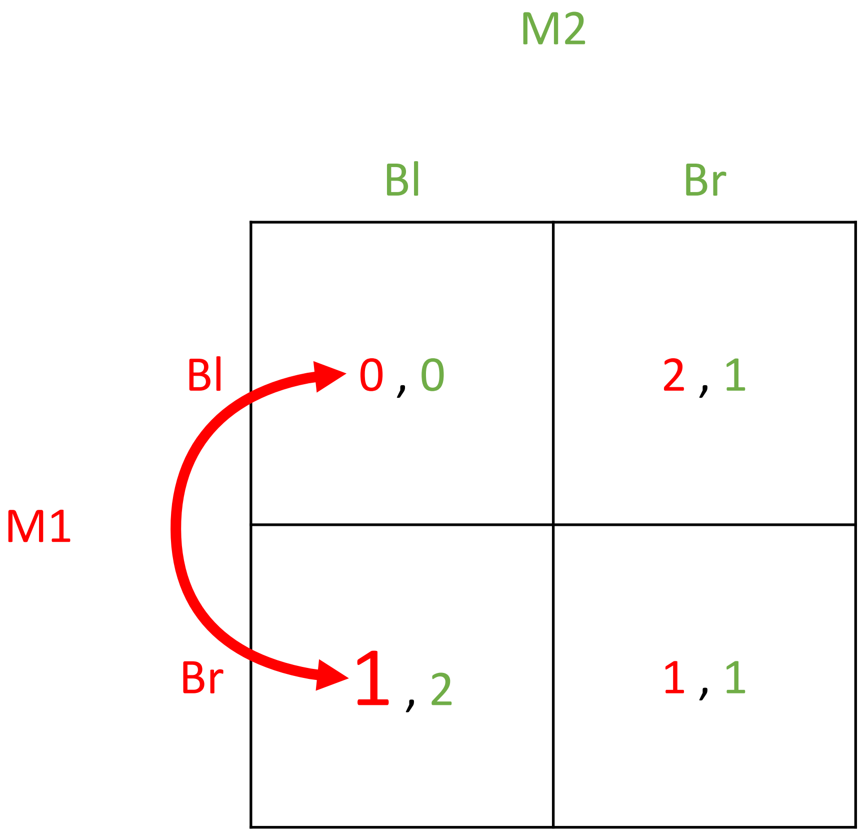
Nash Blonde Game

- 2 or more lusty males
- Several interested females
- At least one more female than male
- Just one female blonde
- Every male prefers blonde to brunette and brunette to no companion

Nash Blonde Game – normal form

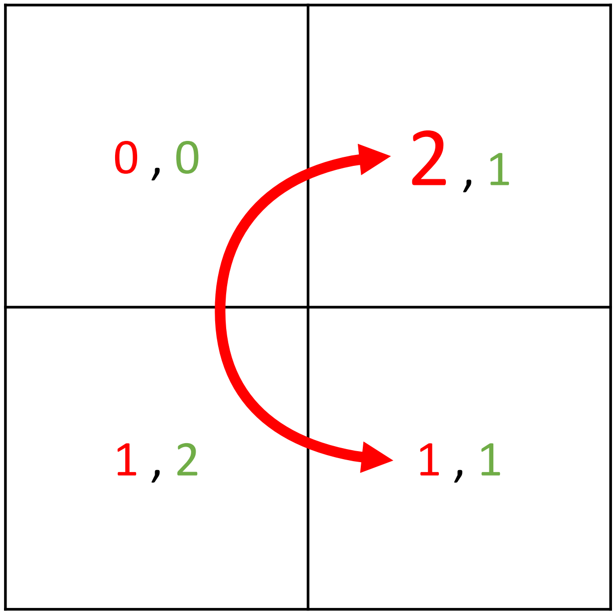
		M2	
		Bl	Br
M1	Bl	0 , 0	2 , 1
	Br	1 , 2	1 , 1

Nash Blonde Game – normal form



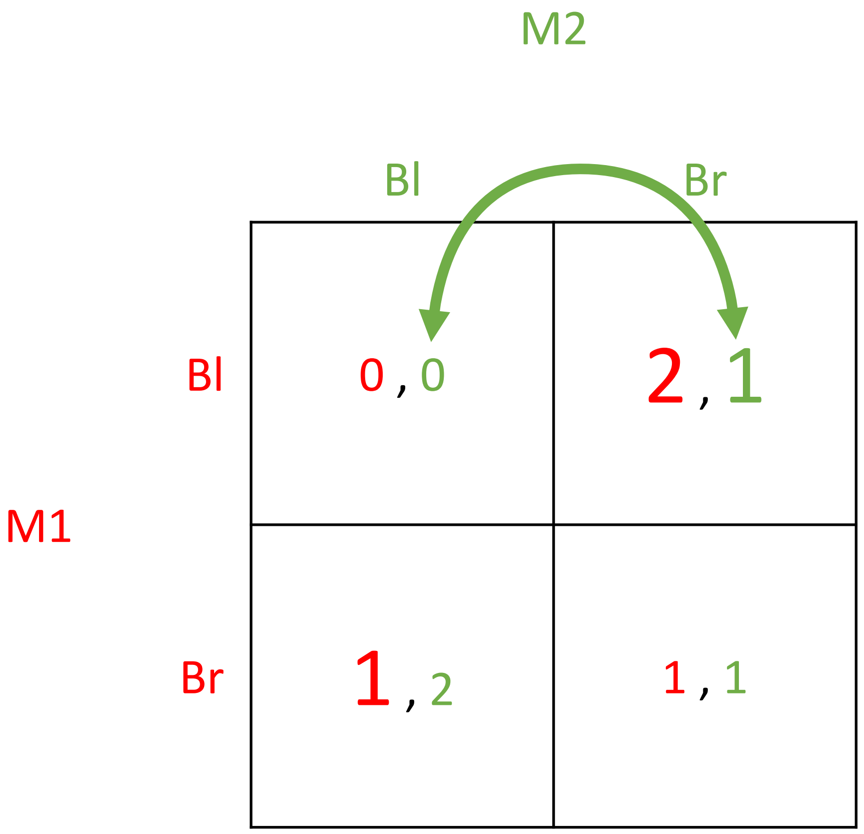
		M2	
		Bl	Br
M1	Bl	0 , 0	2 , 1
	Br	1 , 2	1 , 1

Nash Blonde Game – normal form



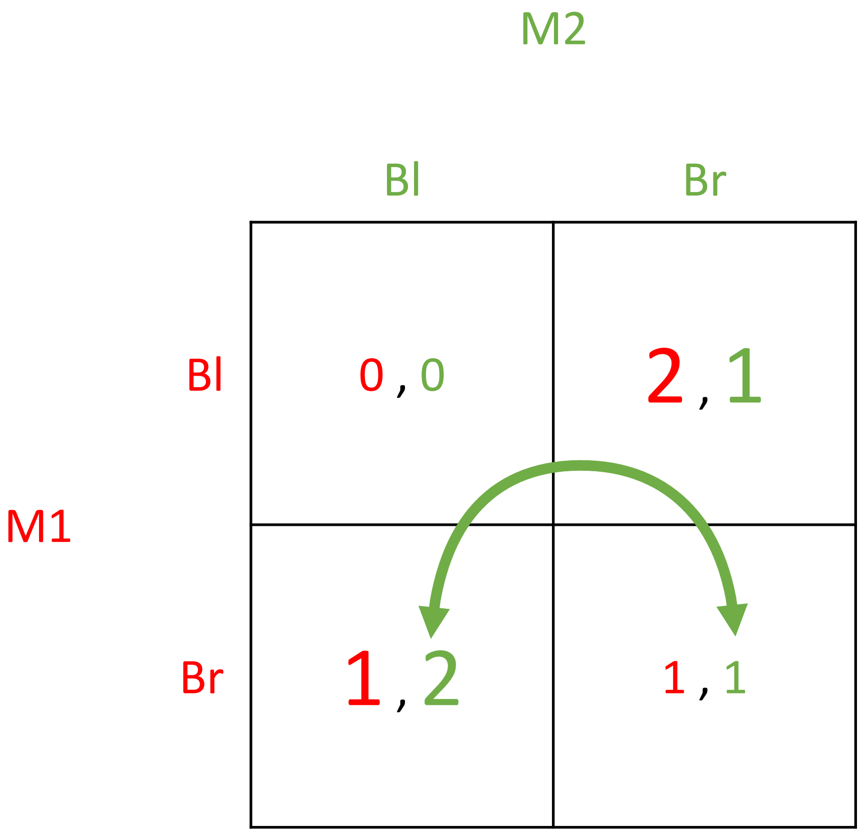
		M2	
		Bl	Br
M1	Bl	0 , 0	2 , 1
	Br	1 , 2	1 , 1

Nash Blonde Game – normal form



		M2	
		Bl	Br
M1	Bl	0 , 0	2 , 1
	Br	1 , 2	1 , 1

Nash Blonde Game – normal form



M2

		Bl	Br
M1	Bl	0 , 0	2 , 1
	Br	1 , 2	1 , 1

Nash Blonde Game – normal form

		M2	
		Bl	Br
M1	Bl	0 , 0	2 , 1
	Br	1 , 2	1 , 1

Nash Equilibrium

- Set of strategies, one for each player, such that **no player has incentive to unilaterally change** her action

- Players are in equilibrium if a change in strategies by any one of them would lead player to earn less (considering strategies of others') than if she remained with her current strategy

- **Mutual best response** to **others' choices**

A

		L	C	R
	T	1 , 1	0 , 0	0 , 0
B	M	0 , 2	1 , 1	2 , -1
	B	0 , 0	1 , 2	2 , 1

A

		L	C	R
	T	1 , 1	0 , 0	0 , 0
B	M	0 , 2	1 , 1	2 , -1
	B	0 , 0	1 , 2	2 , 1

			A	
		L	C	R
	T	1 , 1	0 , 0	0 , 0
B	M	0 , 2	1 , 1	2 , -1
	B	0 , 0	1 , 2	2 , 1

		A		
		L	C	R
B	T	1 , 1	0 , 0	0 , 0
	M	0 , 2	1 , 1	2 , -1
	B	0 , 0	1 , 2	2 , 1

Games might have more NE

Pure strategy equilibrium

- Two equilibriums in this game
- (T , L)
    - u(A) = 1
    - u(B) = 1

- (C , B)
    - u(A) = 1
    - u(B) = 2

- These are **pure strategy equilibriums**

Other basic games

Chicken

		B	
		s	h
A	S	5 , 5	0 , 10
	H	10 , 0	-10 , -10

Chicken NE

- Pure strategies NE
  - (H , s)
    - EU(A) = 10
    - EU(B) = 0
  - (S , h)
    - EU(A) = 0
    - EU(B) = 10
- Mixed strategies NE
  - (1/2 S , 1/2 s)
    - EU(A) = 5/2
    - EU(B) = 5/2

		B	
		s	h
A	S	5 , 5	0 , 10
	H	10 , 0	-10 , -10

Stag hunt

		B	
		s	r
A	S	5 , 5	0 , 3
	R	3 , 0	3 , 3

Stag hunt NE

- Pure strategies NE
  - (S , s)
    - EU(A) = 5
    - EU(B) = 5
  - (R , r)
    - EU(A) = 3
    - EU(B) = 3
- Mixed strategies NE
  - (3/5 S , 3/5 s)
    - EU(A) = 3
    - EU(B) = 3

B

A

	s	R
S	5 , 5	0 , 3
R	3 , 0	3 , 3

B

A

	s	r
S	5 , 5	0 , 7
R	7 , 0	3 , 3

Stag hunt

		B	
		s	r
A	S	5 , 5	0 , 3
	R	3 , 0	3 , 3

Mixed-strategy Nash equilibrium

Matching pennies

- Two players
- Players choose heads or tails
- If players match heads/tails, I (Player 1) win both coins
- If players don’t match heads/tails, opponent (Player 2) wins both coins

Matching pennies

		My pair	
		Heads	Tails
Me	Heads	1 , -1	-1 , 1
	Tails	-1 , 1	1 , -1

Matching pennies – mixed strategy

My pair

		Heads (0.5)	Tails (0.5)
Me	Heads (0.5)	1 , -1	-1 , 1
	Tails (0.5)	-1 , 1	1 , -1

Calculation of mixed-strategy NE

Modified Matching pennies

B

A

	L (q)	R (1 - q)
U (p)	3 , -3	-2 , 2
D (1 - p)	-1 , 1	0 , 0

Mixed strategy game – Player A

- Player A plans to mix Up and Down strategy at a certain ratio
- Player B might play Left or Right
- Player A must find such a probability of playing U and D that makes Player B indifferent to selecting L or R
- Player B has to gain same utility from B’s choice Left and Right
  - $EU_L = EU_R$

- Expected utility of Player B chosing Left:
  - $EU_L = f(p)$

- Expected utility of Player B chosing Right:
  - $EU_R = f(p)$

MS game - Player A’s strategy

- $EU_L = f(p)$
- Some % of time (p) gets B utility -3
- Rest of the time (1 - p) gets B utility 1

- $EU_L = (p)*(-3) + (1 - p)*(1)$
- $EU_L = -3p + 1 - p$
- $EU_L = 1 - 4p$

		B	
		L (q)	R (1 - q)
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

MS game - Player A’s strategy

- $EU_R = f(p)$
- Some % of time (p) gets B utility 2
- Rest of the time (1 - p) gets B utility 0

- $EU_R = (p)*(2) + (1 - p)*(0)$
- $EU_R = 2p + 0 - 0p$
- $EU_R = 2p$

B

		L (q)	R (1 - q)
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

Comparison of EU_L with EU_R

- $EU_L = 1 - 4p$
- $EU_R = 2p$

- $EU_L \qquad = EU_R$
- $1 - 4p \qquad = 2p \qquad +4p$
- $1 \qquad = 6p \qquad /6$
- $\mathbf{p \qquad = 1/6}$

- **1 - p** = 1 - 1/6 = **5/6**

- We've found the ideal mixed strategy for Player A
- If Player A plays Up 1/6 of time and Down 5/6 of time, Player B is indifferent to choosing Left or Right
- We need to do the same for player B

MS game - Player B’s strategy

- $EU_U = f(q)$
- Some % of time (q) gets A utility 3
- Rest of the time (1 - q) gets A utility -2

- $EU_U = (q)*(3) + (1 - q)*(-2)$
- $EU_U = 3q - 2 + 2q$
- $EU_U = 5q - 2$

B

		L (q)	R (1 - q)
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

MS game - Player B’s strategy

- $EU_D = f(q)$
- Some % of time (q) gets A utility -1
- Rest of the time (1 - q) gets A utility 0

- $EU_D = (q)*(-1) + (1 - q)*(0)$
- $EU_D = -1q + 0 - 0q$
- $EU_D = -q$

		B	
		L (q)	R (1 - q)
A	U (p)	3 , -3	-2 , 2
	D (1 - p)	-1 , 1	0 , 0

Comparison of EU_U with EU_D

- $EU_U = 5q - 2$
- $EU_D = -q$

- $EU_U \quad = EU_D$
- $5q - 2 \quad = -q \qquad -5q$
- $-2 \quad = -6q \qquad /-6$
- $\mathbf{q \quad = 1/3}$

- $\mathbf{1 - q} = 1 - 1/3 = \mathbf{2/3}$

- We’ve found the ideal mixed strategy for Player B
- If Player B plays Left 1/3 of time and Down 2/3 of time, Player A is indifferent to choosing Up or Down

Mixed strategy NE

(1/6 U , 1/3 L)

Battle of sexes

- Want to go out together but have no means of communication
- Have 2 choices – ballet or box fight
- Player A prefers box fight
- Player B prefers ballet
- Both prefer being together than being alone

- Preferences for player A: F > B > A
- Preferences for player B: B > F > A

Battle of sexes

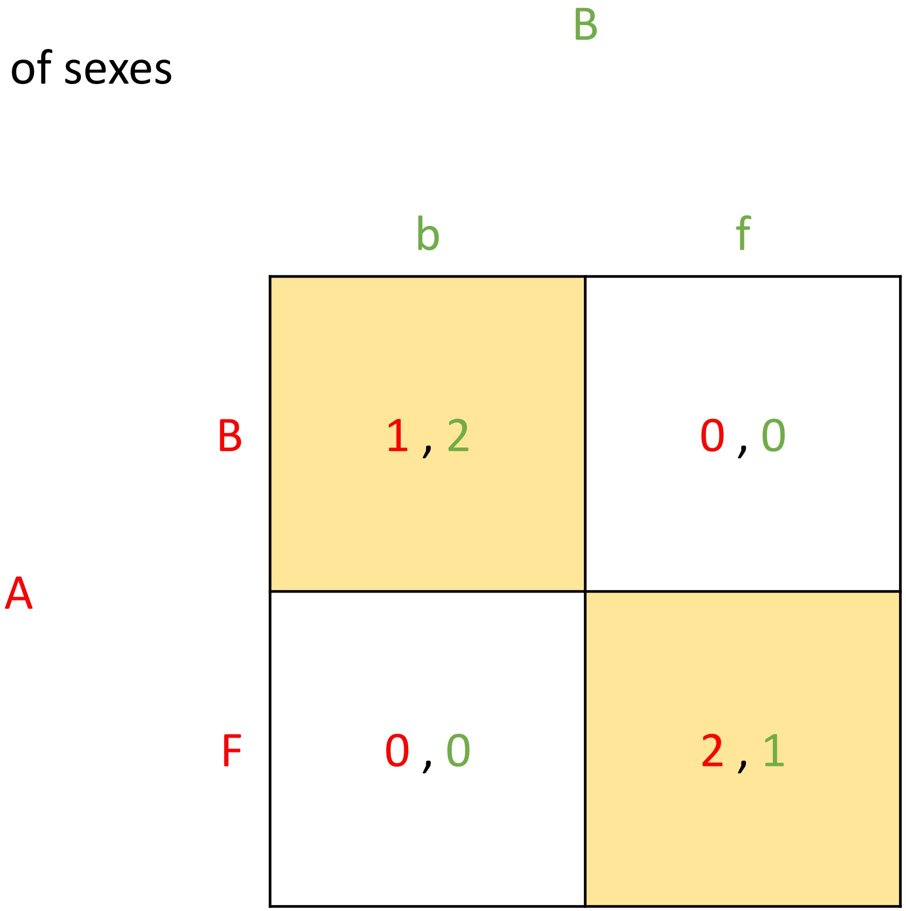
Battle of sexes

B

		b	f
A	B	1 , 2	0 , 0
	F	0 , 0	2 , 1

Battle of sexes – PS equilibriums

Battle of sexes



		B	
		b	f
A	B	1 , 2	0 , 0
	F	0 , 0	2 , 1

Equilibriums

- 2 pure-strategies equilibriums
- How would they coordinate?
- Apart from pure strategies equilibriums there is one mixed strategy equilibrium for this game
- (1/3 B, 2/3 b)