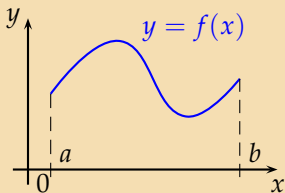


Délka rovinné křivky

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Délka rovinné křivky $y = f(x)$ $x \in \langle a, b \rangle$, která je na intervalu $\langle a, b \rangle$ diferencovatelná.



$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Vypočtete délku oblouku křivky $y = \ln \sin x$ na intervalu $\langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle$.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

Zderivujeme funkci.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x} \right)^2} dx$$

Dosadíme derivaci do vzorce pro délku křivky.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx$$

Upravíme na společného jmenovatele.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

Zjednodušíme.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$\cos x = t$$

=

Budeme integrovat goniometrickou funkci, $\sin x$ je v liché mocnině, proto použijeme substituci $\cos x = t$. Musíme tedy zlomek přepsat do vhodného tvaru.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} =$$

$$\cos x = t$$

$$-\sin x dx = dt$$

Do čitatele se snažíme vzhledem k substituci dostat $\sin x dx$. Rozšíříme proto zlomek $\sin x$:

$$\frac{1}{\sin x} = \frac{\sin x}{\sin^2 x} = \frac{\sin x}{1 - \cos^2 x}$$

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} =$$

$$\begin{aligned} \cos x &= t \\ -\sin x dx &= dt \\ t_1 &= \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 &= \cos \frac{2\pi}{3} = -\frac{1}{2} \end{aligned}$$

Při dosazení substituce budeme také potřebovat najít meze nové proměnné.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} =$$

$$\begin{aligned} \cos x &= t \\ -\sin x dx &= dt \\ t_1 &= \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 &= \cos \frac{2\pi}{3} = -\frac{1}{2} \end{aligned}$$

$$= \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

Dosadíme.

$$y = \ln \sin x, \quad x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, \quad L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} =$$

$$\begin{aligned} \cos x &= t \\ -\sin x dx &= dt \\ t_1 &= \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 &= \cos \frac{2\pi}{3} = -\frac{1}{2} \end{aligned}$$

$$= \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1-t^2}$$

$$= \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{-\frac{1}{2}}$$

Najdeme primitivní funkci.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

$$\begin{aligned} \cos x &= t \\ -\sin x dx &= dt \\ t_1 &= \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 &= \cos \frac{2\pi}{3} = -\frac{1}{2} \end{aligned}$$

$$= \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{1}{2} \ln \left| \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} \right| + \frac{1}{2} \ln \left| \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right|$$

Dosadíme meze.

$$y = \ln \sin x, x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

$$\begin{aligned} \cos x &= t \\ -\sin x dx &= dt \\ t_1 &= \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 &= \cos \frac{2\pi}{3} = -\frac{1}{2} \end{aligned}$$

$$= \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{1}{2} \ln \left| \frac{1-\frac{1}{2}}{1+\frac{1}{2}} \right| + \frac{1}{2} \ln \left| \frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right| = -\frac{1}{2} \ln \left| \frac{1}{3} \right| + \frac{1}{2} \ln 3$$

Zjednodušíme zlomky v argumentech logaritmov.

$$y = \ln \sin x, \quad x \in \left\langle \frac{\pi}{3}, \frac{2\pi}{3} \right\rangle, \quad L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x dx}{1 - \cos^2 x} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

| |
|------------------------------------------|
| $\cos x = t$ |
| $-\sin x dx = dt$ |
| $t_1 = \cos \frac{\pi}{3} = \frac{1}{2}$ |

$$\ln \frac{1}{3} = -\ln 3, \text{ proto } -\frac{1}{2} \ln \frac{1}{3} + \frac{1}{2} \ln 3 = \frac{1}{2} \ln 3 + \frac{1}{2} \ln 3 = \ln 3.$$

$$= \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{1}{2} \ln \left| \frac{1-\frac{1}{2}}{1+\frac{1}{2}} \right| + \frac{1}{2} \ln \left| \frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right| = -\frac{1}{2} \ln \left| \frac{1}{3} \right| + \frac{1}{2} \ln 3$$

$$= \ln 3$$

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