MATEMATIKA PRE EMBRYOLÓGOV

cvičenie 9)

priebeh funkcie

príklad 1. Vyšetrite priebeh funkcie $f(x) = \frac{1}{1+x^2}$.

- $D(f) = \mathbb{R}$ (môžeme dosadiť čokoľvek)

$f(-x) = \frac{1}{1+(-x)^2} = \frac{1}{1+x^2} = f(x) \Rightarrow$ f je párna

f nie je nepárna, nie je periodická (kým v predpise nie je goniometrická funkcia, periodicita nemá ako vzniknúť)

priesečníky s osou y: $x=0$, $f(0)=1 \longrightarrow [0,1]$

priesečníky s osou x: $y=0$, $f(x)=0$ - to nenastane nikdy $\Rightarrow$ nemá priesečníky s osou x

$f(x) > 0$ pre každé $x \in D(f) = \mathbb{R}$, f je všade kladná

- $$f'(x) = \frac{(1)' \cdot (1+x^2) - 1 \cdot (1+x^2)'}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}$$

stacionárne body: $f'(x) = 0 \Rightarrow -\frac{2x}{(1+x^2)^2} = 0 \Leftrightarrow -2x = 0$
$\Leftrightarrow x = 0$

	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	−
$f(x)$	↗	↘

$f'(x)$	+		−
$f(x)$	↗	0	↘

$\Rightarrow$ v bode $x=0$ má f lokálne maximum

f na intervale $(-\infty, 0)$ rastie

f na intervale $(0, \infty)$ klesá

- $$f''(x) = \frac{(-2x)' \cdot (1+x^2)^2 - (-2x)\left((1+x^2)^2\right)'}{(1+x^2)^4} =$$

$$= \frac{-2 \cdot (1+x^2)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} = \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3} =$$

$$= \frac{6x^2 - 2}{(1+x^2)^3}$$

$$f''(x) = 0 \iff \frac{6x^2-2}{(1+x^2)^3} = 0 \iff 6x^2 - 2 = 0$$

$$6x^2 = 2$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$$

	$\left(-\infty, -\frac{\sqrt{3}}{3}\right)$	$\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$	$\left(\frac{\sqrt{3}}{3}, \infty\right)$
$f''(x)$	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{+}{+} = +$
$f(x)$	$\cup$	$\cap$	$\cup$

v tomto riadku dosadíme do $f''(x)$ nejaký bod z uvedených intervalov

$\Rightarrow \pm\frac{\sqrt{3}}{3}$ sú inflexné body

f je na intervale $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ konvexná

— || — $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ konkávna

— || — $\left(\frac{\sqrt{3}}{3}, \infty\right)$ konvexná

$$f\left(\pm\frac{\sqrt{3}}{3}\right) = \frac{1}{1+\left(\pm\frac{\sqrt{3}}{3}\right)^2} = \frac{3}{4}$$

- asymptoty

asymptoty bez smernice funkcia f nemá lebo $D(f) = \mathbb{R}$ a nemá žiadne body nespojitosti

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{1}{x^3 + x} = \frac{1}{\pm\infty} = 0$$

$$\lim_{x \to \pm\infty} (f(x) - 0x) = \lim_{x \to \pm\infty} \frac{1}{x^2+1} = \frac{1}{\infty} = 0$$

poznámka $y=0$ je asymptotou so smernicou v $\pm\infty$

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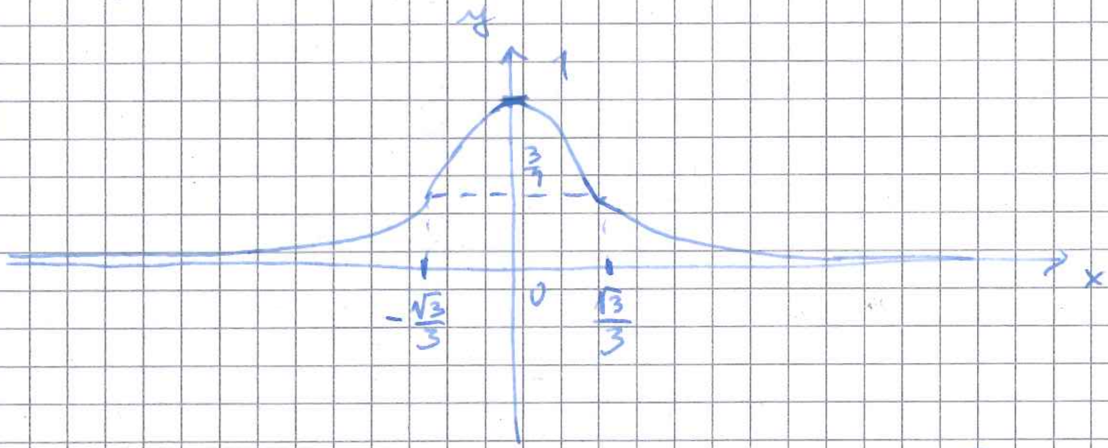


príklad 2.) Vyšetrite priebeh funkcie $f(x) = \dfrac{-x^2}{x+1}$

- $D(f) = \mathbb{R} \setminus \{-1\}$ Keďže $1 \in D(f)$, ale $-1 \notin D(f)$, funkcia f nie je ani párna, ani nepárna.

f nie je periodická

priesečníky s osou y: $x=0$, $f(0)=0 \rightsquigarrow [0,0]$

priesečníky s osou x: $y=0$, $f(x)=0$

$$-\frac{x^2}{x+1}=0 \iff x^2=0 \iff x=0 \rightsquigarrow [0,0]$$

	$(-\infty,-1)$	$(-1,0)$	$(0,\infty)$
$f(x)$	$\frac{-}{-} = +$	$\frac{-}{+} = -$	$\frac{-}{+} = -$

f je na intervale $(-\infty,-1)$ kladná

f je na intervale $(-1,\infty)$ nekladná

- $$f'(x) = \frac{(-x^2)'(1+x) - (-x^2)(1+x)'}{(1+x)^2} = \frac{-2x(1+x)+x^2}{(1+x)^2} =$$

$$= \frac{-x^2-2x}{(1+x)^2}$$

stacionárne body: $f'(x)=0 \iff \dfrac{-x^2-2x}{(1+x)^2}=0$

$$\iff x^2+2x=0$$

$$x(x+2)=0$$

stacionárne body sú $x = 0, -2$

	$(-\infty, -2)$	$(-2, -1)$	$(-1, 0)$	$(0, \infty)$
$f'(x)$	$\frac{-}{+} = -$	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{-}{+} = -$
$f(x)$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$

v bode -2 má f lokálne minimum

v bode 0 má f lokálne maximum

$$f(-2) = \frac{-4}{-2+1} = 4$$

f na intervale $(-\infty, -2)$ klesá

$(-2, -1)$ rastie

$(-1, 0)$ rastie

$(0, \infty)$ klesá

- $\cancel{f''(x) = \frac{(-x^2-2x)'(1+x^2) - (-x^2-2x)(1+x^2)'}{(1+x^2)^4} =}$

$\cancel{= \frac{(-2x-2)(1+x^2) + (x^2+2x)\cdot 2x}{(1+x^2)^4} = \frac{2x^2-2x-2}{(1+x^2)^4}}$

$$f''(x) = \frac{(-x^2-2x)'(1+x)^2 - (-x^2-2x)\left((1+x)^2\right)'}{(1+x)^4} =$$

$$= \frac{(-2x-2)(1+x)^2 + (x^2+2x)\cdot 2\cdot(1+x)}{(1+x)^4} = \frac{(-2x-2)(1+x) + 2(x^2+2x)}{(1+x)^3}$$

$$= -\frac{2}{(1+x)^3}$$

$f''(x) = 0$ nemá riešenie

	$(-\infty, -1)$	$(-1, \infty)$
$f''(x)$	$+$	$-$
$f(x)$	$\cup$	$\cap$

f je na $(-\infty, -1)$ konvexná

f je na $(-1, \infty)$ konkávna

- asymptoty

asymptoty bez smernice:

$$\lim_{x \to (-1)^-} f(x) = \lim_{x \to (-1)^-} -\frac{x^2}{1+x} = -\frac{((-1)^-)^2}{1+(-1)^-} = -\frac{1^+}{0^-} = \infty$$

$$\lim_{x \to (-1)^+} f(x) = \lim_{x \to (-1)^+} -\frac{x^2}{1+x} = -\frac{((-1)^+)^2}{1+(-1)^+} = -\frac{1^-}{0^+} = -\infty$$

$\Rightarrow$ $x = -1$ je asymptotou bez smernice

asymptoty so smernicou:

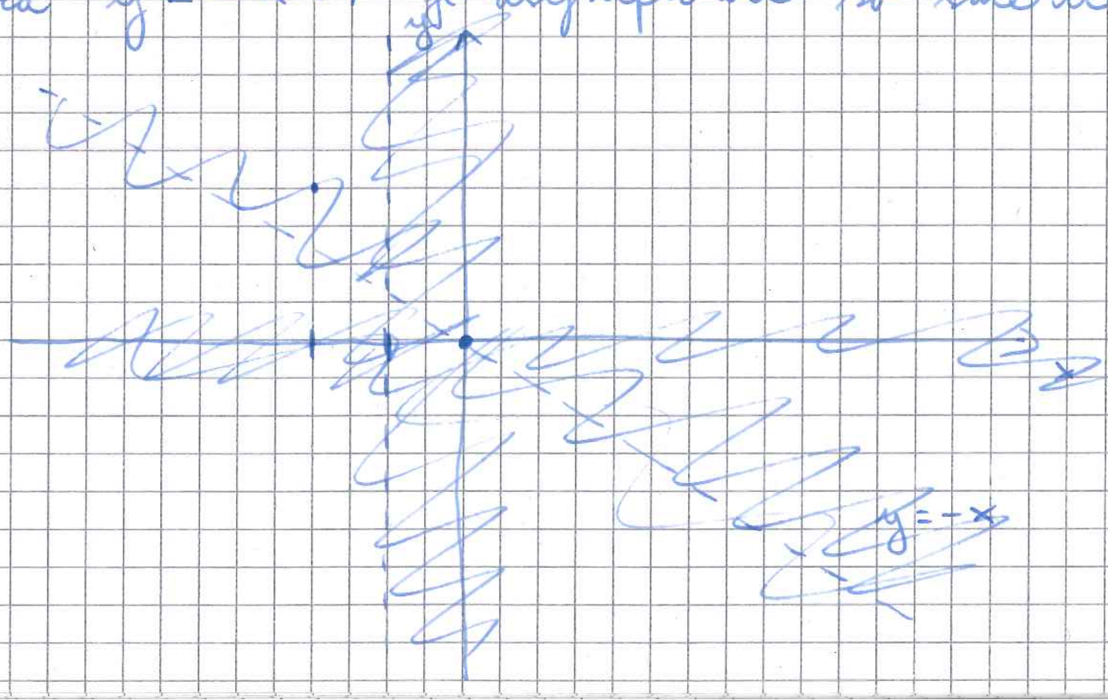
$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} -\frac{x}{1+x} = -\lim_{x \to \pm\infty} \frac{1}{1+\frac{1}{x}} = -1$$

$$q = \lim_{x \to \pm\infty} (f(x) - kx) = \lim_{x \to \pm\infty} \left(-\frac{x^2}{1+x} + x\right) = \lim_{x \to \pm\infty} \frac{-x^2+x+x^2}{1+x} =$$

$$= \lim_{x \to \pm\infty} \frac{x}{1+x} = 1$$

priamka $y = -x+1$ je asymptotou so smernicou

~~graf~~



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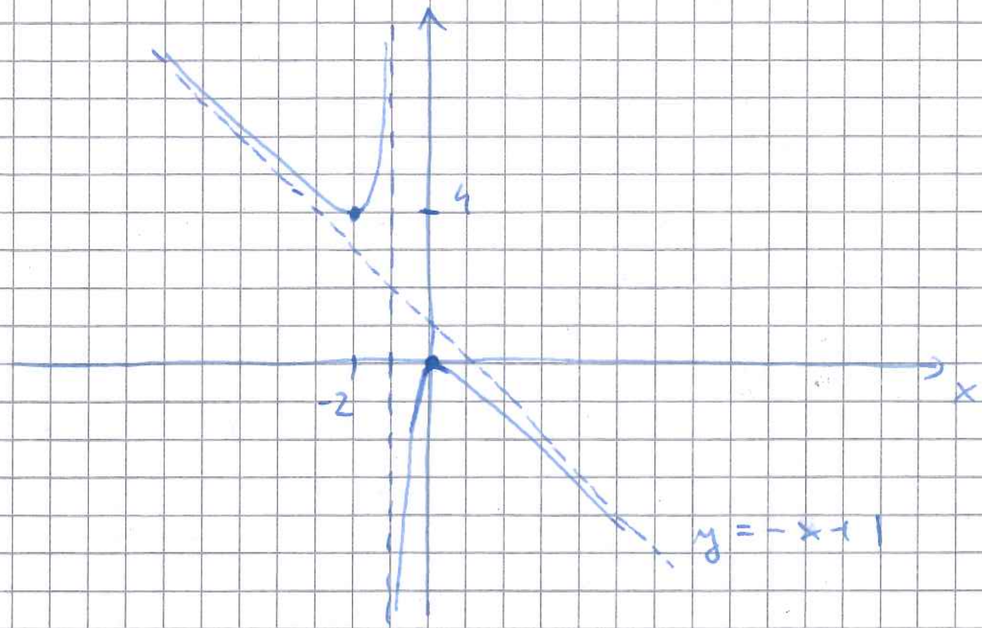


príklad 3.) Vyšetrite priebeh funkcie $f(x) = x - \operatorname{arctg} x$.

$D(f) = \mathbb{R}$

$f(-x) = -x - \operatorname{arctg}(-x) = -x + \operatorname{arctg} x = -(x - \operatorname{arctg} x)$
$= -f(x)$

↓

arctg x je nepárna funkcia

$\Rightarrow$ f je nepárna funkcia

f nie je párna, f nie je periodická

priesečníky s osou y: $x = 0$, $f(0) = 0$ $\rightsquigarrow [0,0]$

priesečníky s osou x: $y = 0$ $f(x) = 0$

$x - \operatorname{arctg} x = 0$

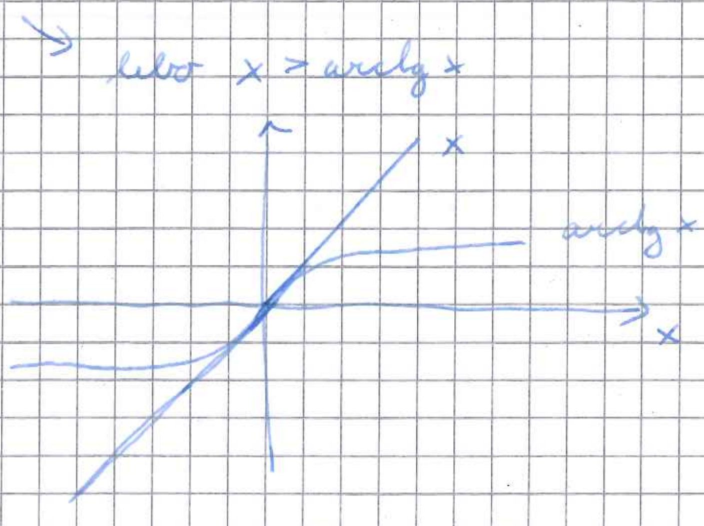
$x = \operatorname{arctg} x \Leftrightarrow x = 0$

$\rightsquigarrow [0,0]$

	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	$-$	$+$

↓ lebo $x < \operatorname{arctg} x$

$\Rightarrow$ lebo $x > \operatorname{arctg} x$



- $f'(x) = (x - \operatorname{arctg} x)' = 1 - \frac{1}{1+x^2} = \frac{1+x^2-1}{1+x^2} = \frac{x^2}{1+x^2}$

Stacionárne body: $f'(x) = 0$

$$\frac{x^2}{1+x^2} = 0 \iff x^2 = 0 \iff x = 0$$

	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$+$	$+$
$f(x)$	$\nearrow$	$\nearrow$

$\Rightarrow$ f rastie na celom $\mathbb{R}$

$\Rightarrow$ nemá lokálne extrémy

- $f''(x) = \left(\frac{x^2}{1+x^2}\right)' = \cancel{2x}\ \frac{(x^2)'(1+x^2) - x^2(1+x^2)'}{(1+x^2)^2} =$

$$= \frac{2x(1+x^2) - x^2 \cdot 2x}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

$$f''(x) = 0 \iff \frac{2x}{(1+x^2)^2} = 0 \iff 2x = 0 \iff x = 0$$

	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	$\frac{-}{+} = -$	$\frac{+}{+} = +$
$f(x)$	$\frown$	$\smile$

f je na $(-\infty, 0)$ konkávna

f je na $(0, \infty)$ konvexná

0 je inflexný bod

- asymptoty

f nemá asymptoty bez smernice

$$\lim_{x \to \pm\infty} \frac{x - \operatorname{arctg} x}{x} = \lim_{x \to \pm\infty} \left(1 - \frac{\operatorname{arctg} x}{x}\right) =$$

$$= 1 - \lim_{x \to \pm\infty} \left(\underbrace{\frac{1}{x}}_{\to 0} \cdot \underbrace{\operatorname{arctg} x}_{\text{ohraničená}}\right) = 1$$

~~$\lim_{x\to\pm\infty} (f(x)-x) = \lim$~~

$$\lim_{x\to\infty} (f(x)-x) = \lim_{x\to\infty} (x-\operatorname{arctg}x - x) = \lim_{x\to\infty} (-\operatorname{arctg}x) = -\frac{\pi}{2}$$

$$\lim_{x\to-\infty} (f(x)-x) = \lim_{x\to-\infty} (x-\operatorname{arctg}x - x) = \lim_{x\to-\infty} (-\operatorname{arctg}x) = \frac{\pi}{2}$$

~~asy~~ priamka $y = x - \frac{\pi}{2}$ je asymptota v $+\infty$

$y = x + \frac{\pi}{2}$ je asymptota v $-\infty$

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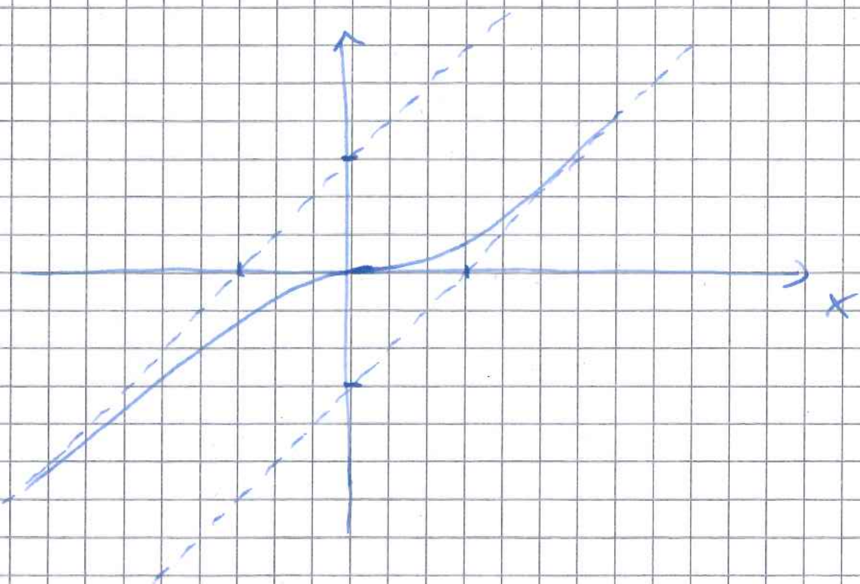


príklad 4.) Vyšetrite priebeh funkcie $f(x) = \frac{x^3}{x^2-1}$.

$D(f) = \mathbb{R} \setminus \{\pm 1\}$

$$f(-x) = \frac{(-x)^3}{(-x)^2-1} = \frac{-x^3}{x^2-1} = -\frac{x^3}{x^2-1} = -f(x)$$

$\Rightarrow$ f je nepárna (platí aj $x \in D(f) \Rightarrow -x \in D(f)$)

f nie je párna, f nie je periodická

$$f(x) = 0 \Leftrightarrow \frac{x^3}{x^2-1} = 0 \Leftrightarrow x^3 = 0 \Leftrightarrow x = 0$$

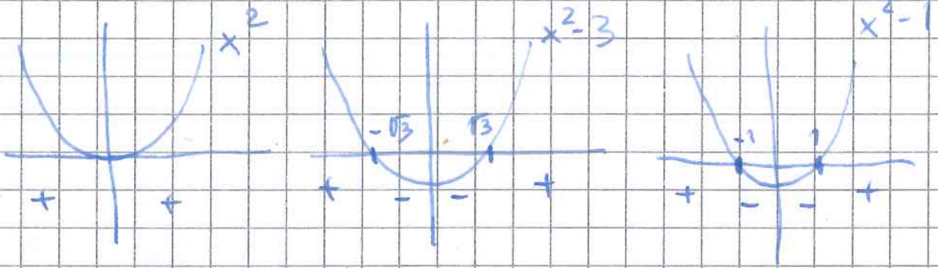
	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$f(x)$	$\frac{-}{+} = -$	$\frac{-}{-} = +$	$\frac{+}{-} = -$	$\frac{+}{+} = +$

priesečník s osami je len jeden: $[0, 0]$

- $$f'(x) = \left(\frac{x^3}{x^2-1}\right)' = \frac{(x^3)'(x^2-1) - x^3(x^2-1)'}{(x^2-1)^2} =$$

$$= \frac{3x^2(x^2-1) - x^3 \cdot 2x}{(x^2-1)^2} = \frac{x^4-3x^2}{(x^2-1)^2}$$

~~$f(x)=$~~ stacionárne body: $f'(x)=0 \iff \frac{x^4-3x^2}{(x^2-1)^2} = 0$



$\iff x^4 - 3x^2 = 0$

$\iff x^2(x^2-3) = 0$

$\iff x = 0 \vee x = \pm\sqrt{3}$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$\frac{+\cdot+}{+} = +$	$\frac{+\cdot-}{+} = -$	$\frac{+\cdot-}{+} = -$	$\frac{+\cdot-}{+} = -$	$\frac{+\cdot-}{+} = -$	$\frac{+\cdot+}{+} = +$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$

v bode $-\sqrt{3}$ je lokálne maximum

$$f(-\sqrt{3}) = \frac{(-\sqrt{3})^3}{(-\sqrt{3})^2-1} = \frac{-3\cdot\sqrt{3}}{2}$$

v bode $\sqrt{3}$ je lokálne minimum

$$f(\sqrt{3}) = \frac{3\sqrt{3}}{2}$$

f je nepárna

- $$f''(x) = \left(\frac{x^4-3x^2}{(x^2-1)^2}\right)' = \frac{(x^4-3x^2)'(x^2-1)^2 - (x^4-3x^2)\left((x^2-1)^2\right)'}{(x^2-1)^4} =$$

$$= \frac{(4x^3-6x)(x^2-1)^2 - (x^4-3x^2)\cdot 2\cdot(x^2-1)\cdot 2x}{(x^2-1)^4} =$$

$$= \frac{(4x^3-6x)(x^2-1) - 4x(x^4-3x^2)}{(x^2-1)^3} = \frac{2x^3+6x}{(x^2-1)^3}$$

$$f''(x) = 0 \iff \frac{2x^3+6x}{(x^2-1)^3} = 0 \iff 2x^3+6x = 0$$

$$\iff 2x(x^2+3) = 0$$

$$\iff x = 0$$

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$f''(x)$	$\frac{-}{+} = -$	$\frac{-}{-} = +$	$\frac{+}{-} = -$	$\frac{+}{+} = +$
$f(x)$	$\cap$	$\cup$	$\cap$	$\cup$

bod 0 je inflexný bod

- asymptoty

– asymptoty bez smernice:

$$\cancel{\lim_{x\to-\infty} \frac{x^3}{x^2-1} = \lim_{x\to-\infty} \frac{x}{1-\frac{1}{x^2}} = -\infty}$$

$$\cancel{\lim_{x\to\infty} \frac{x^3}{x^2-1} = \lim_{x\to\infty} \frac{x}{1-\frac{1}{x^2}}}$$

$$\lim_{x\to(-1)^-} \frac{x^3}{x^2-1} = \frac{((-1)^-)^3}{((-1)^-)^2-1} = \frac{(-1)^-}{1^+-1} = \frac{(-1)^-}{0^+} = -\infty$$

$$\lim_{x\to(-1)^+} \frac{x^3}{x^2-1} = \frac{((-1)^+)^3}{((-1)^+)^2-1} = \frac{(-1)^+}{1^--1} = \frac{(-1)^+}{0^-} = \infty$$

priamka $x = -1$ je asymptotou bez smernice

$$\lim_{x\to1^-} \frac{x^3}{x^2-1} = \frac{(1^-)^3}{(1^-)^2-1} = \frac{1^-}{1^--1} = \frac{1^-}{0^-} = -\infty$$

$$\lim_{x\to1^+} \frac{x^3}{x^2-1} = \frac{(1^+)^3}{(1^+)^2-1} = \frac{1^+}{1^+-1} = \frac{1^+}{0^+} = \infty$$

priamka $x = 1$ je asymptotou bez smernice

- asymptoty so smernicou

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{\frac{x^3}{x^2-1}}{x} = \lim_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1$$

$$q = \lim_{x \to \pm\infty} (f(x) - kx) = \lim_{x \to \pm\infty} \left(\frac{x^3}{x^2-1} - x \right) =$$

$$= \lim_{x \to \pm\infty} \frac{x^3 - x^3 + x}{x^2-1} = \lim_{x \to \pm\infty} \frac{x}{x^2-1} = \lim_{x \to \pm\infty} \frac{1}{x - \frac{1}{x}} = 0$$

$y = x$ je asymptota so smernicou v $+\infty$ aj $-\infty$

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