príklad:

$$\int_0^1 x^2 \operatorname{arccotg} x \, dx$$

$$\int x^2 \operatorname{arccotg} x \, dx = \left| \begin{matrix} u = \operatorname{arccotg} x & u' = -\frac{1}{1+x^2} \\ v' = x^2 & v = \frac{x^3}{3} \end{matrix} \right|$$

$$= \frac{x^3}{3} \operatorname{arccotg} x + \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx = \frac{x^3}{3} \operatorname{arccotg} x +$$

$$+ \frac{1}{3} \int \frac{x(1+x^2) - x}{1+x^2} \, dx = \frac{x^3}{3} \operatorname{arccotg} x + \frac{1}{3} \int x \, dx$$

$$+ - \frac{1}{3} \int \frac{x}{1+x^2} \, dx = \frac{x^3}{3} \operatorname{arccotg} x + \frac{x^2}{6} - \frac{1}{6} \int \frac{2x}{1+x^2} \, dx$$

$$= \underbrace{\frac{x^3}{3} \operatorname{arccotg} x + \frac{x^2}{6} - \frac{1}{6} \log|1+x^2| + c}_{= F(x)}$$

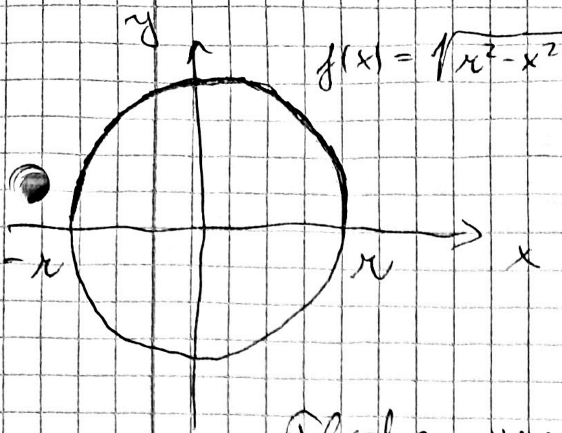
$= F(x)$ túto primitívnu funkciu použijeme v Newtonovej - Leibnizovej formule

$$\int_0^1 x^2 \operatorname{arccotg} x \, dx = \left[F(x) \right]_0^1 = F(1) - F(0) =$$

$$= \frac{\pi}{12} + \frac{1}{6} - \frac{\log(2)}{6} - \left(0 + 0 - 0 \right) = \frac{\pi + 2 - \log 4}{12}$$

príklad: Pomocou ~~k~~ určitého integrálu odvoďte vzorce pre výpočet obvodu a obsahu kruhu a vzorce pre výpočet objemu a povrchu gule.

Riešenie: Kružnica so stredom v bode $[0,0]$ s polomerom r je daná rovnicou $y^2 + x^2 = r^2$.



$f(x) = \sqrt{r^2 - x^2}$

$$y^2 + x^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$|y| = \sqrt{r^2 - x^2}$$
$$y = \pm\sqrt{r^2 - x^2}$$

Plocha vrchného polkruhu je $\int_{-r}^{r} f(x)\,dx$.

1. postup: najprv spočítame neurčitý integrál, a potom určitý pomocou Newtonovej-Leibnizovej formule.

$$\int \sqrt{r^2 - x^2}\,dx \;\left|\; \begin{array}{l} x = r\sin t \Rightarrow t = \arcsin\frac{x}{r} \\ dx = r\cos t\,dt \end{array} \right| =$$

$$= \int \sqrt{r^2 - r^2\sin^2 t}\cdot r\cos t\,dt = \int \sqrt{r^2(1-\sin^2 t)}\cdot r\cos t\,dt$$

$$= \int r\cdot\sqrt{1-\sin^2 t}\cdot r\cos t\,dt = r^2 \int \underbrace{\sqrt{1-\sin^2 t}}_{\text{goniometrická jednotka}} \cos t\,dt$$

$$= r^2 \int \cos^2 t\,dt$$

ďalej potrebujeme nasledujúci trik: $\cos(2t) = \cos^2 t - \sin^2 t$

$$= \cos^2 t - (1 - \cos^2 t)$$
$$= 2\cos^2 t - 1$$

$$= r^2 \int \frac{\cos(2t) + 1}{2}\,dt$$

$$= \frac{r^2}{2}\int \cos(2t) + 1\,dt \qquad \Rightarrow \cos^2 t = \frac{\cos(2t)+1}{2}$$

$$= \frac{r^2}{2}\left(\frac{\sin(2t)}{2} + t + c\right) = \frac{r^2}{2}\left(\frac{\sin\left(2\arcsin\left(\frac{x}{r}\right)\right)}{2} + \arcsin\frac{x}{r}\right) + c$$

už len stačí použiť NL formulu:

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \left[\frac{r^2}{2} \left(\frac{\sin(2 \arcsin \frac{x}{r})}{2} + \arcsin \frac{x}{r} \right) \right]_{-r}^{r}$$

$$= \frac{r^2}{2} \left(\frac{\sin \pi}{2} + \frac{\pi}{2} - \left(\frac{\sin(-\pi)}{2} - \frac{\pi}{2} \right) \right) =$$

$$= \frac{\pi r^2}{2} \quad \ldots \text{ Obsah kruhu je } \pi r^2$$

2. postup: Metódy per-partes a substitučné metódy sa dajú použiť aj pre určitý integrál, stačí, keď pri ich použití prepočítame hranice (substitúcia) alebo funkciu $u \cdot v$ vyhodnotíme $[u \cdot v]_b^a$ (per-partes)

Tu sú všeobecné vzorčeky:

$$\int_a^b u(x) v'(x)\, dx = \underbrace{[u(x) v(x)]_a^b}_{} - \int_a^b u'(x) v(x)\, dx$$

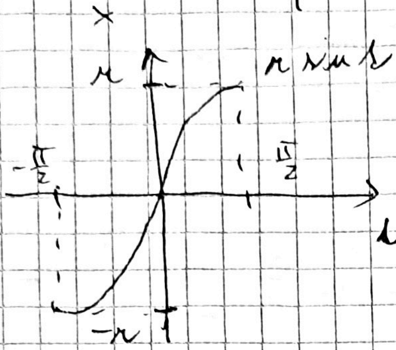
$$= u(b) v(b) - u(a) v(a)$$

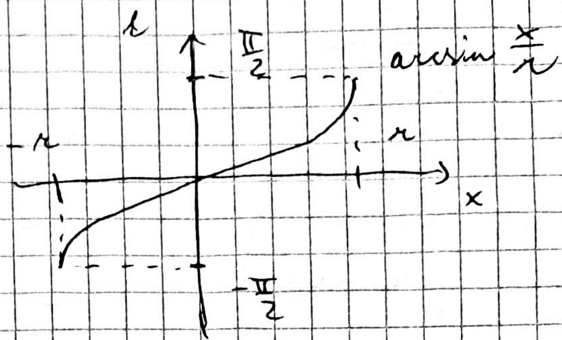
$$\int_a^b f(\varphi(t)) \cdot \varphi'(t)\, dt = \left| \begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t)\, dt \\ b \rightsquigarrow \varphi(b) \\ a \rightsquigarrow \varphi(a) \end{array} \right| = \int_{\varphi(a)}^{\varphi(b)} f(x)\, dx$$

$$\int_a^b f(t)\, dt \left| \begin{array}{l} t = \psi(x) \Rightarrow x = \psi^{-1}(t) \\ dt = \psi'(x)\, dx \\ b \rightsquigarrow \psi^{-1}(b) \\ a \rightsquigarrow \psi^{-1}(a) \end{array} \right| = \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} f(\psi(x))\, \psi'(x)\, dx$$

Transformácie φ, ψ musia byť na daných intervaloch prosté (musia byť rastúce alebo klesajúce).

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \left| \begin{array}{l} x = r \sin t \Rightarrow t = \arcsin \frac{x}{r} \\ dx = r \cos t \, dt \\ r \rightsquigarrow \arcsin \frac{r}{r} = \arcsin 1 = \frac{\pi}{2} \\ -r \rightsquigarrow \arcsin\left(-\frac{r}{r}\right) = \arcsin(-1) = -\frac{\pi}{2} \end{array} \right|$$





$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 t} \cdot r \cos t \, dt = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t}\, \cos t \, dt$$

$$= r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(2t) + 1}{2} \, dt =$$

NL form. $$= \frac{r^2}{2} \left[\frac{\sin(2t)}{2} + t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{r^2}{2} \left(\frac{\sin \pi}{2} + \frac{\pi}{2} - \left(\frac{\sin(-\pi)}{2} - \frac{\pi}{2} \right) \right)$$

$$= \frac{\pi r^2}{2} \ldots$$

obvod kruhu: dĺžka hornej polkružnice sa rovná dĺžke grafu funkcie $f(x) = \sqrt{r^2 - x^2}$ na intervale $[-r, r]$, čo môžeme spočítať takto:

$$\int_{-r}^{r} \sqrt{1 + f'(x)^2}\, dx$$

$$f'(x) = \left(\sqrt{r^2 - x^2}\right)' = \left(\left(r^2 - x^2\right)^{\frac{1}{2}} \right)' = \frac{1}{2} \cdot \left(r^2 - x^2\right)^{-\frac{1}{2}} \cdot (-2x)$$

$$= -\frac{x}{\sqrt{r^2 - x^2}}$$

$$\int_{-r}^{r} \sqrt{1+f'(x)^2}\, dx = \int_{-r}^{r} \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx =$$

$$= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\, dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\, dx = r\int_{-r}^{r} \frac{1}{\sqrt{r^2-x^2}}\, dx$$

$$= \left(\begin{array}{l}\text{vzorec +}\\ \text{NL formula}\end{array}\right) = r\left[\arcsin\frac{x}{r}\right]_{-r}^{r} =$$

$$= r\left(\arcsin 1 - \arcsin(-1)\right) = r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi r$$

--- obsod kroga je $2\pi r$

volumen krogle: krogla vrtilne rotacije (pod) grafa funkcije $f(x) = \sqrt{r^2-x^2}$ okoli osi x na intervalu $[-r, r]$, če merimo spodaj tako:

$$\pi \int_{-r}^{r} f^2(x)\, dx = \pi \int_{-r}^{r} \left(\sqrt{r^2-x^2}\right)^2 dx = \pi \int_{-r}^{r} r^2 - x^2\, dx$$

$$= \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \pi\left(r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right) =$$

$$= \cancel{\pi}\ \frac{4}{3}\pi r^3$$

površina krogle: $2\pi \int_{-r}^{r} f(x)\sqrt{1+f'(x)^2}\, dx =$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2-x^2} \cdot \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx = 2\pi \int_{-r}^{r} \sqrt{r^2-x^2} \cdot \sqrt{\frac{r^2}{r^2-x^2}}\, dx$$

$$= 2\pi \int_{-r}^{r} r\, dx = 2\pi\left[rx\right]_{-r}^{r} = 2\pi\left(r^2 - (-r^2)\right) = 4\pi r^2$$